Extensions to a Theorem of Barndorff-Neilsen and Schou

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Abstract. The theorem of Barndorff-Neilsen and Schou (1973, Theorem 2) stating that the admissible region of autoregressive process may be obtained as image of the admissible region defined by the partial autocorrelations under the Durbin-Levinson recursion is extended to the case of the boundary and to subset autoregressions. These results may be used to obtain a visualization of the admissible region for AR(4) and higher order autoregressions. Additionally these results suggest a new approach to subset autoregressive modelling.

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Running head: Extensions BNS Theorem

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1. INTRODUCTION

The AR(p) model with mean zero may be written, $\phi(B)z_t = a_t$, where $\phi(B) = 1 - \phi_1 B - \ldots \phi_p B^p$ and a_t is Gaussian white noise with variance σ_a^2. The admissible region for stationary-causal processes is defined by the region in $\Re^p$ for which all roots of $\phi(B) = 0$ lie outside the unit circle (Brockwell and Davis, 1991).

Let $\phi_{j,k}$ denote the coefficient of z_{t-j} in the minimum-mean-square-error predictor of z_t given $z_{t-1}, \ldots, z_{t-k}$. Then the partial autocorrelations, denoted by ζ_i for $i = 1, \ldots, p$ are given by $\zeta_i = \phi_{i,i}$, where the $\phi_{i,i}$ are determined by the Durbin-Levinson recursion,

$$\phi_{j,k+1} = \phi_{j,k} - \phi_{k+1,k+1}\phi_{k+1-j,k}, \quad j = 1, \ldots, k \tag{1}$$

where $k = 1, \ldots, p$. Barndorff-Neilsen and Schou (1973) showed that eq. (1) can be used to define a transformation, $\mathcal{B} : (\zeta_1, \ldots, \zeta_p) \Rightarrow (\phi_1, \ldots, \phi_p)$, and that this transformation is one-to-one, continuous and differentiable inside the admissible region. Monahan (1984) derived an algorithm for computing $\mathcal{B}^{-1}$. It will be convenient in §3, to use the notation $\phi_i = \mathcal{B}_i(\zeta)$ to refer to each parameter $i = 1, \ldots, p$, where $\zeta = (\zeta_1, \ldots, \zeta_p)$. It should be noted that since the determinant Jacobian of the transformation (Barndorff-Neilsen and Schou, 1973, p.414) is zero on the boundary, the transformation is not 1:1 there. For example, in the AR(2) case $\mathcal{B}(\zeta_1, \zeta_2) = (\zeta_1(1-\zeta_2), \zeta_2)$ so all the points on the boundary of the form $(\zeta_1, 1)$ are mapped into the point $(0, 1)$.

2. BOUNDARY OF AR(p) ADMISSIBLE REGION

Let $\mathcal{D}_\phi$ denote the admissible region for an AR(p) in the parameter space $(\phi_1, \ldots, \phi_p)$. The admissible region for the transformed parameters $(\zeta_1, \ldots, \zeta_p)$ is the interior of the unit cube,

$$\mathcal{D}_\zeta = \{(\zeta_1, \ldots, \zeta_p) \in \Re^p : |\zeta_i| < 1, i = 1, \ldots, p\}. \tag{2}$$

Denote the boundary sets of $\mathcal{D}_\phi$ and $\mathcal{D}_\zeta$ by ∂_ϕ and ∂_ζ. Theorem 2 of Barndorff-Nielsen and Schou (1973) may be stated, $\mathcal{D}_\phi = \mathcal{B}(\mathcal{D}_\zeta)$. For visualization it is the boundary region that it is of interest and Theorem 1 below extends the result of Barndorff-Nielsen and Schou (1973, Theorem 2) to the boundary.

THEOREM 1. $\partial_\phi = \mathcal{B}(\partial_\zeta)$

PROOF. First we show that $\mathcal{B}(\partial_\zeta) \subset \partial_\phi$. Let $\bar{\mathcal{D}}_\phi$ and $\bar{\mathcal{D}}_\zeta$ denote the closures of $\mathcal{D}_\phi$ and $\mathcal{D}_\zeta$ respectively. So $\bar{\mathcal{D}}_\phi = \mathcal{D}_\phi \cup \partial_\phi$ and $\mathcal{D}_\phi \cap \partial_\phi = \emptyset$. Similarly, $\bar{\mathcal{D}}_\zeta = \mathcal{D}_\zeta \cup \partial_\zeta$ and $\mathcal{D}_\zeta \cap \partial_\zeta = \emptyset$. Since $\mathcal{B}$ is a polynomial and hence continuous, $\mathcal{B}(\bar{\mathcal{D}}_\zeta) = \mathcal{D}_\phi \cup \mathcal{B}(\partial_\zeta) \subset \overline{\mathcal{B}(\mathcal{D}_\zeta)} = \bar{\mathcal{D}}_\phi = \mathcal{D}_\phi \cup \partial_\phi$. Since $\mathcal{D}_\phi \cap \mathcal{B}(\partial_\zeta) = \emptyset$, $\partial_\phi \supset \mathcal{B}(\partial_\zeta)$.

Next, we show $\partial_\phi \subset \mathcal{B}(\partial_\zeta)$. Let $\varphi \in \partial_\phi$. Then there exists a sequence of distinct elements $\varphi_n \in \mathcal{D}_\phi$ such that $\varphi_n \to \varphi$. Let $\vartheta_n = \mathcal{B}^{-1}(\varphi_n)$. Then ϑ_n are distinct elements and there exists a subsequence $\{\vartheta_{n_k}\} \subset \{\vartheta_n\}$ such that $\vartheta_{n_k} \to \vartheta \in \bar{\mathcal{D}}_\zeta$. Since $\mathcal{B}$ is continuous on $\bar{\mathcal{D}}_\zeta$, $\mathcal{B}(\vartheta_{n_k}) \to \mathcal{B}(\vartheta)$. Since $\vartheta \in \partial_\zeta$, $\mathcal{B}(\vartheta) \in \mathcal{B}(\partial_\zeta)$. Hence, $\partial_\phi \subset \mathcal{B}(\partial_\zeta)$. $\square$

The application of Theorem 1 is illustrated for the AR(4) model, $z_t = \phi_1 z_{t-1} + \phi_2 z_{t-2} + \phi_3 z_{t-3} + \phi_4 z_{t-4}$. In this case the admissible region is 4-dimensional. However we can explore this surface by examining the 3D surfaces corresponding to a fixed value of the parameter ϕ_4. It may be shown that,

$$\begin{aligned}
\phi_1 &= \zeta_1 - \zeta_1\,\zeta_2 - \zeta_2\,\zeta_3 - \zeta_3\,\zeta_4, \\
\phi_2 &= \zeta_2 - \zeta_1\,\zeta_3 + \zeta_1\,\zeta_2\,\zeta_3 - \zeta_2\,\zeta_4 + \zeta_1\,\zeta_3\,\zeta_4 - \zeta_1\,\zeta_2\,\zeta_3\,\zeta_4, \\
\phi_3 &= \zeta_3 - \zeta_1\,\zeta_4 + \zeta_1\,\zeta_2\,\zeta_4 + \zeta_2\,\zeta_3\,\zeta_4 \\
\phi_4 &= \zeta_4.
\end{aligned} \tag{3}$$

It follows from Theorem 1 that the admissible surface can be plotted in 3D space for a fixed value of ϕ_4 by varying two of the ζ-parameters at a time over their admissible range while holding the third parameter fixed at ± 1. An example 3D plot is shown in Figure 1 below for the AR(4) with $\phi_4 = 0.5$.

[Figure 1]

3. EXTENSION TO SUBSET AR

Subset autoregressive provide a parsimonious alternative to the full AR(p) model which is especially useful in modelling seasonal or periodic time series. Consider the subset AR(4) model, $z_t = \phi_1 z_{t-1} + \phi_4 z_{t-4}$. Then the transformation,

$$\mathcal{B}^{-1}(\phi_1, 0, 0, \phi_4) = (\zeta_1, \zeta_2, \zeta_3, \zeta_4) \tag{4}$$

is one-to-one, continuous and differentiable in the admissible region. However the admissible region in the four-dimensional space $\mathcal{D}_\zeta$ is no longer simply the interior unit 4D-cube. Instead it is a complicated two-dimensional subspace of the 4D-cube. Setting the $\phi_2 = 0$ and $\phi_3 = 0$ in the second and third equations in (3) these equations may be solved for ζ_2 and ζ_3 to determine $\mathcal{B}$. When $\zeta_1 \neq 0$, $\zeta_4 \neq 0$, $|\zeta_1| < 1$ and $\zeta_4 > -1$,

$$\begin{aligned} \zeta_2 &= \frac{1 + 2\,{\zeta_1}^2\,\zeta_4 - \sqrt{1 + 4\,{\zeta_1}^2\,\zeta_4\,(1+\zeta_4)}}{2\left(-1+{\zeta_1}^2\right)\,\zeta_4} \\ \zeta_3 &= \frac{-1 + \sqrt{1 + 4\,{\zeta_1}^2\,\zeta_4\,(1+\zeta_4)}}{2\,\zeta_1\,(1+\zeta_4)}. \end{aligned} \tag{5}$$

Note that not all solutions of (5) are admissible. For example taking $\zeta_1 = 0.9$ and $\zeta_4 = -0.9$, (5) produces, $\zeta_2 \doteq -0.88$ and $\zeta_3 \doteq -3.8$. Since (4) is also a one-to-one, continuous and differentiable transformation, these equations establish that the transformation $(\phi_1, \phi_4) \to (\zeta_1, \zeta_4)$ is one-to-one, continuous and differentiable with the appropriate admissible regions. The admissible region for ζ_1 and ζ_4 is given by,

$$\mathcal{D}_\zeta = \{(\zeta_1, \zeta_4) \in (-1,1) \times (-1,1) : |\zeta_2| < 1 \wedge |\zeta_3| < 1\} \tag{6}$$

The admissible region for ζ_1 and ζ_4 is shown in Figure 3 below. Applying the transformation $\mathcal{B} : (\zeta_1, \zeta_2, \zeta_3, \zeta_4) \to (\phi_1, 0, 0, \phi_4)$, the admissible region, Figure 4, for ϕ_1 and ϕ_4 is obtained.

[Figures 2 and 3]

In the general case of a subset AR(p) model, let

$$z_t = a_t + \phi_{i_1} z_{t-i_1} + \ldots + \phi_{i_m} z_{t-i_m} \tag{7}$$

where $i_1 < \ldots < i_m$. Let $\dot{\mathcal{B}}^{-1}_{i_1,\ldots,i_m}$ denote the transformation,

$$\dot{\mathcal{B}}^{-1}_{i_1,\ldots,i_m} : (\phi_{i_1}, \ldots, \phi_{i_m}) \to (\zeta_{i_1}, \ldots, \zeta_{i_m}) \tag{8}$$

obtained by selecting the $i_1, \ldots, i_m$ elements from,

$$\mathcal{B}^{-1} : (\phi_1, \ldots, \phi_p) \to (\zeta_1, \ldots, \zeta_p). \tag{9}$$

THEOREM 2. The transformation defined in eqn. (8) is one-to-one, continuous and differentiable in the admissible region.

PROOF. The transformation given in eqn. (1) may be written,

$$\phi_i = \zeta_i + b_i(\zeta_1, \ldots, \zeta_p), \tag{10}$$

where $b_i(\zeta_1, \ldots, \zeta_p)$ is a multilinear function of the variables $\zeta_1, \ldots, \zeta_p$ with the coefficient of each term being 1 or -1. This holds for $p = 1$. Suppose it is true also for $p = k$. Then eqn. (1) ensures that the new values of $\phi_1, \ldots, \phi_{k+1}$ are suitable multilinear functions of the variables $\zeta_1, \ldots, \zeta_{k+1}$.

Consider a subset AR(p) with only one parameter $\phi_h = 0$ constrained, where h is a fixed value $0 < h \leq p$. Using (10), an explicit solution of the equation $\phi_h = 0$ may be found,

$$\zeta_h = g_h(\zeta^{(-h)}), \tag{11}$$

where $\zeta^{(-h)}$ denotes the vector $(\zeta_1, \ldots, \zeta_p)$ with the h-th element removed. We can subsitute for ζ_h in each of the $p - 1$ equations $\phi_i = \mathcal{B}_i(\zeta_1, \ldots, \zeta_p)$ to obtain an explicit solution $\phi_i = \mathcal{B}_{i,h}(\zeta^{(-h)})$ which defines the required transformation.

In the more general case, there are $p - m$ constraints. Denote the indices of those parameters $\phi_1, \ldots \phi_p$ which are constrained to zero by $j_1, \ldots, j_{p-m}$. The $p - m$ constraint equations in $p - m$ unknowns may be written $0 = \zeta_{j_k} + b_{j_k}(\zeta_1, \ldots, \zeta_p), k = 1, \ldots, (p - m)$. Starting with the equation corresponding to $\phi_{j_1} = 0$, the solution for ζ_{j_1} can be found and substituted in the remaining $p-1$ equations. The next equation, $\phi_{j_2} = 0$, is by the multilinear property, a quadratic with two solutions for ζ_{j_2}. Continuing this process, there are $(p-m)!$ solutions to the constraint equations. The theorem of Barndorff-Neilsen and Schou (1973) guarantees that at most only one of these solutions is admissible. □

Denote the admissible boundaries for the parameters $(\phi_{i,1}, \ldots, \phi_{i,m})$ and $(\zeta_{i,1}, \ldots, \zeta_{i,m})$ by $\partial_{\phi_{i,1},\ldots,\phi_{i,m}}$ and $\partial_{\zeta_{i,1},\ldots,\zeta_{i,m}}$ respectively.

THEOREM 3. $\partial_{\phi_{i,1},\ldots,\phi_{i,m}} = \dot{\mathcal{B}}_{i_1,\ldots,i_m}(\partial_{\zeta_{i,1},\ldots,\zeta_{i,m}})$

PROOF. Since $\dot{\mathcal{B}}_{i_1,\ldots,i_m}$ is a one-to-one, continuous and differentiable transformation, Theorem 3 follows making the necessary changes to the argument given by Theorem 1. □

The application of Theorem 3 was illustrated by the derivation of Figure 4.

Theorem 2 indicates that the parameter space of the subset autoregression is spanned by the corresponding partial autocorrelations. This suggests an alternative parameterization for the subset autoregessive model in which some partial autocorrelations are set to zero and the others are estimated. A subset modelling approach based on this idea is developed in our forthcoming article (McLeod and Zhang, 2000a) where it is shown that this approach has many advantages over the usual subset modelling approach.

Mathematica notebooks for generating all figures in this article and for visualizing the AR(4) admissible region in more detail are available from Mcleod and Zhang (2000b).

REFERENCES

Barndorff-Nielsen, O. and Schou G. (1973). On the Parametrization of Autoregressive Models by Partial Autocorrelations *Journal of Multivariate Analysis* 3, 408-419.

Brockwell, P.J. and Davis, R.A. (1991). *Time Series: Theory and Methods*. (2nd edn.). New York: Springer-Verlag.

McLeod, A.I. and Zhang, Y. (2002a). Electronic supplement to "Extensions to a Theorem of Barndorff-Neilsen and Schou" http://www.stats.uwo.ca/faculty/aim/epubs/AR4/.

McLeod, A.I. and Zhang, Ying (2002b). Subset Autoregression: A New Approach. Submitted for publication.

Monahan, J.F. (1984). A note on enforcing stationarity in autoregressive-moving average models, *Biometrika* 71, 403-404.

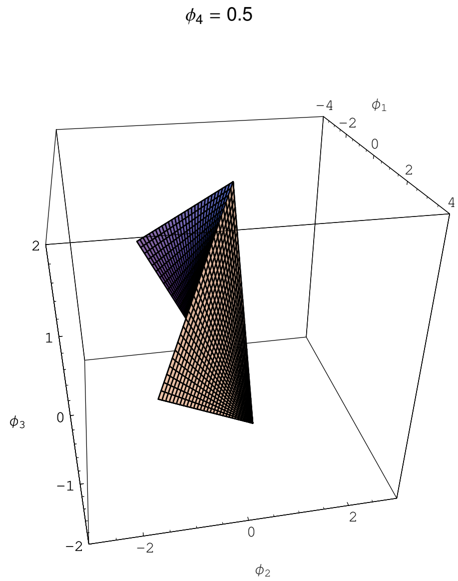

Figure 1: Admissible region, $z_t = \phi_1 z_{t-1} + \phi_2 z_{t-2} + \phi_3 z_{t-3} + \phi_4 z_{t-4}$, with $\phi_4 = 0.5$.

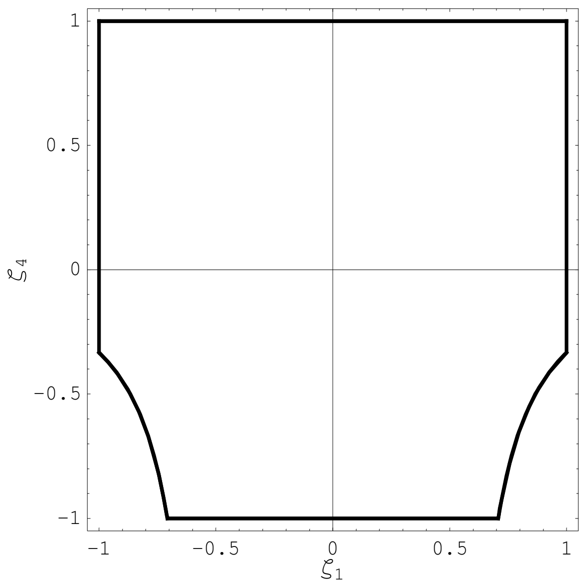

Figure 2: Admissible region, $z_t = \phi_1 z_{t-1} + \phi_4 z_{t-4}$ for reparameterized model with ζ_1 and ζ_4.

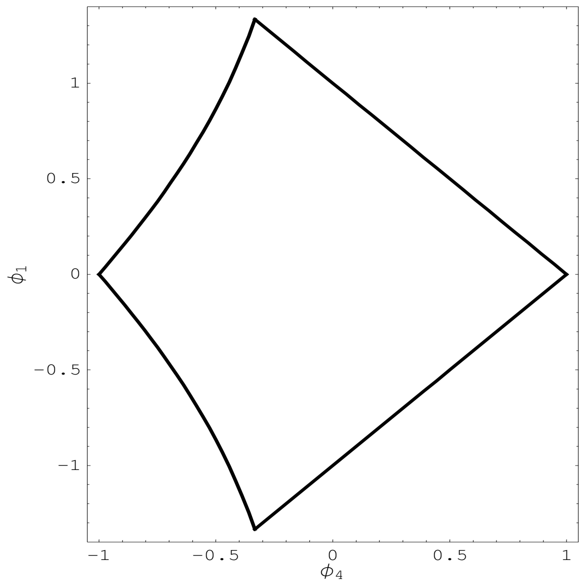

Figure 3: Admissible region, $z_t = \phi_1 z_{t-1} + \phi_4 z_{t-4}$.