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Consistent detection of a monotonic trend superposed on a stationary time series

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SUMMARY

Consider a time series made up of a signal and a stationary autocorrelated error series. A statistic is proposed for examining the hypothesis that the signal term is constant versus the hypothesis that it is monotonic in time. The statistic is the ratio of a linear combination of the time series values, with coefficients introduced by Abelson & Tukey (1963), to an estimate of the standard error of the linear combination. The statistic has asymptotic power 1 for a broad class of monotonic alternatives. The procedure is illustrated for the series of river heights at a location on the Rio Negro in Brazil where there is concern that the height is rising due to deforestation of the Amazon Basin. The significance level obtained is 0·025.

Some key words: Amazon Basin; Asymptotic methods; Central limit theorem; Change; Consistent test; Monotonic trend; Power; Rio Negro; Stationary time series.

1. INTRODUCTION

1·1. *Preamble*

In the study of a scientific phenomenon via time series data, a fundamental question that sometimes arises is: is there a trend in the series? In particular one may mention the case of the thickness of the atmosphere's ozone layer possibly decreasing with increasing use of chlorofluorocarbons (Stolarski, 1988), and the case of the long-term height of the Amazon river increasing with deforestation (Sternberg, 1987). If there are changes in these two cases, it seems reasonable to view the changes as monotonic. These are particular examples of circumstances where a stimulus applied is increasing with time, and so there could well be a monotonic response effect.

To begin, consider the time series model

$$Y(t)=S(t)+E(t), \tag{1·1}$$

for $t=0, \pm 1, \pm 2, \ldots,$ where $S(.)$ is a deterministic signal and $E(.)$ is a zero mean stationary noise series. Of interest is the hypothesis: $S(t)$ equal constant, versus the alternative: $S(t) \leqslant S(t+1)$ for all t, with strict inequality for some t.

Given data $Y(t)$ $(t=0, \ldots, T-1)$, the procedure studied in this paper is based on a linear combination, $\Sigma\, c(t)\, Y(t)$, involving particular coefficients $c(.)$. The advantage of employing such a linear combination is that an estimate of its variance is directly available.

1·2. *Earlier work in the autocorrelated time series case*

Time series researchers have long approached the problem of trend analysis via the technique of fitting a parametric form for $S(.)$ and then examining the parameter estimates obtained. For example, one might fit $S(t) = \alpha + \beta t$ and consider the hypothesis $\beta = 0$. Grenander (1954) has worked out the asymptotics in this case. Specifically, given the data $Y(0), \ldots, Y(T-1)$, his methods lead to the result that the statistic

$$\hat{\beta} = \sum_{t=0}^{T-1} (t - \bar{t}) Y(t) \Big/ \sum_{t=0}^{T-1} (t - \bar{t})^2 \tag{1·2}$$

has mean β and asymptotic variance $2\pi f_{EE}(0)/\Sigma\,(t-\bar{t})^2$, where $f_{EE}(\lambda)$ denotes the power spectrum of the series $E(.)$ at frequency λ and $\bar{t} = \frac{1}{2}T - \frac{1}{2}$. Further, under regularity conditions, $\hat{\beta}$ is asymptotically normal, for example, Brillinger (1975, Th. 5.11.1), and so for large T one may compute approximate p-values, confidence intervals and the like.

1·3. *Earlier work in the independent case*

When the error series, $E(.)$, is white noise, that is a sequence of independent identically distributed random variables, quite a number of techniques, both parametric and nonparametric, have been proposed. Lombard (1987) is one recent reference and Shaban (1980) provides a bibliography.

The work for independent errors pertinent to the present problem is that of Abelson & Tukey (1963) who determine a statistic linear in the data and sensitive to monotonic mean function departure, as follows.

Determine coefficients, $c = \{c(t), t = 0, 1, \ldots, T-1\}$, with mean $\bar{c} = 0$ to achieve

$$\max_c \min_S \frac{|\sum \{c(t) - \bar{c}\}\{S(t) - \bar{S}\}|^2}{\sum \{c(t) - \bar{c}\}^2 \sum \{S(t) - \bar{S}\}^2}, \tag{1·3}$$

where

$$S = \{S(0) \leqslant S(1) \leqslant \ldots \leqslant S(T-1)\}.$$

The coefficients (Abelson & Tukey, 1963) are

$$c(t) = c^T(t) = \left\{t\left(1 - \frac{t}{T}\right)\right\}^{\frac{1}{2}} - \left\{(t+1)\left(1 - \frac{t+1}{T}\right)\right\}^{\frac{1}{2}}. \tag{1·4}$$

The value at the extreme for expression (1·3) is $1/\Sigma\, c(t)^2 \simeq 2/\log T$ for large T. This is achieved for the step-function signals, $S(t) = 0$ for $t \leqslant t_0$ and $S(t) = 1$ for $t > t_0$; see Fig. 1

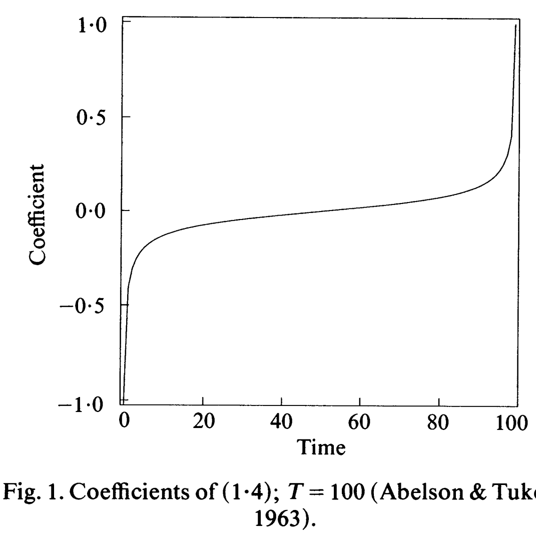

Fig. 1. Coefficients of (1·4); $T = 100$ (Abelson & Tukey, 1963).

for $T=100$. The shape will be the same for other values of T. The linear combination, $\Sigma\, c(t)\, Y(t)$ strongly contrasts the beginning and end levels of the data, as seems intuitively reasonable given that one is looking for a monotonic trend across the time series values.

When the $E(t)$ are independent normals with known variance σ^2, a test statistic is provided by $\Sigma\, c(t)\, Y(t)/\{\sigma^2\, \Sigma\, c(t)^2\}^{\frac{1}{2}}$. In practice if σ^2 is unknown, but one has an independent chi-squared estimate of σ^2, then one can compute a t statistic.

1·4. *Notation and structure of the paper*

Specific assumptions, theorems and proofs have been placed in Appendices. The results are asymptotic, but are hoped to provide useful finite sample approximations. The term 'monotonic' will require inequality of some consecutive values in the sequence. The symbol Σ, unsubscripted, refers to summation over $t=0, \ldots, T-1$. It is assumed that $f_{EE}(0) \neq 0$.

2. The proposed procedure

2·1. *The statistic*

The problem of specific concern in this paper is that of the model (1·1) with an autocorrelated noise series, $E(\cdot)$.

Consider $\Sigma\, c(t)\, Y(t)$ with $Y(t)$ given by (1·1) and with coefficients, $c(t)$, given by (1·4). One has

$$E\{\sum c(t)\, Y(t)\} = \sum c(t) S(t), \tag{2·1}$$

$$\operatorname{var}\{\sum c(t)\, Y(t)\} = \sum_{s,t} c(s) c(t) c_{EE}(s-t), \tag{2·2}$$

where $c_{EE}(u) = \operatorname{cov}\{E(t+u), E(t)\}$. It is shown in Appendix 3 that for large T expression (2·2) is approximately $2\pi f_{EE}(0)\, \Sigma\, c(t)^2$, where $f_{EE}(.)$ is the noise power spectrum. Further if the noise series is normal, then $\Sigma\, c(t)\, Y(t)$ is normal, while if the noise series is mixing, then the linear combination is asymptotically normal; see Appendix 3.

If $\hat{f}_{EE}(0)$ is a consistent estimate of $f_{EE}(0)$, then when $S(t)$ is constant one may approximate the distribution of the statistic

$$\sum c(t)\, Y(t)/\{2\pi \hat{f}_{EE}(0) \sum c(t)^2\}^{\frac{1}{2}}, \tag{2·3}$$

by a standard normal, and thereby compute an approximate p-value or carry out formal tests of significance. The problem of constructing such a consistent estimate of $f_{EE}(0)$ is now addressed.

2·2. *Estimation of the variance*

Suppose that the trend function $S(.)$ has the form $g(t/T)$, where $g(.)$ has a finite Lipshitz integral modulus of continuity; see (A·4) below. The signal is taken to have this form in order that it may be present uniformly the whole of a time interval tending to infinity with T. Let the signal be estimated by the running mean

$$\hat{S}(t) = \sum_{s=-V}^{V} Y(t+s)/(2V+1), \tag{2·4}$$

for $t = V+1, \ldots, T-1-V$ and for moderate V. The noise series may be estimated by the residuals $\hat{E}(t) = Y(t) - \hat{S}(t)$. An estimate of $f_{EE}(0)$ may be based on these last.

Specifically, denote the discrete Fourier transform of the residuals by

$$\hat{\varepsilon}_j = \sum_{t=V+1}^{T-1-V} \hat{E}(t) \exp\left(-\frac{2\pi itj}{T}\right), \tag{2·5}$$

for $j = 0, \ldots, T-1$. Define the transfer function values

$$a_j = \frac{\sin\{2\pi j(2V+1)/2T\}}{(2V+1)\sin(2\pi j/2T)}. \tag{2·6}$$

For chosen L, compute $\hat{f}_{EE}(0)$, the smoothed periodogram spectral estimate

$$\sum_{j=1}^{L} \frac{1}{2\pi T} |\hat{\varepsilon}_j|^2 \Big/ \sum_{j=1}^{L} (1-a_j)^2. \tag{2·7}$$

Here y_j denotes the discrete Fourier transform of the original series. Frequencies from $1/T$ to L/T cycles/unit time are involved, where the latter is to be near 0. The terms $(1-a_j)^2$ are introduced to compensate for the effect of the filtering operation on the noise series.

Specific assumptions are given in the appendices, under which the estimate (2·7) is consistent as $T \to \infty$.

3. POWER

The variate $\Sigma\, c(t) Y(t)$ may be approximated by a normal with mean $\Sigma\, c(t) S(t)$ and with variance $2\pi\hat{f}_{EE}(0) \Sigma\, c(t)^2$. This leads, for example, to approximating a probability such as

$$\text{pr}\,[\sum c(t) Y(t)/\{2\pi\hat{f}_{EE}(0) \sum c(t)^2\}^{\frac{1}{2}} > d], \tag{3·1}$$

for some given d, by

$$1 - \Phi[d - \sum c(t) S(t)/\{2\pi f_{EE}(0) \sum c(t)^2\}^{\frac{1}{2}}], \tag{3·2}$$

where $\Phi[.]$ is the normal cumulative distribution function. Now, from the results of Abelson & Tukey (1963),

$$\frac{|\sum c(t) S(t)|^2}{\sum c(t)^2} \geqslant \frac{2 \sum \{S(t) - \bar{S}\}^2}{\log T} + o(1), \tag{3·3}$$

for all monotonic $S(t)$. It follows that a test based on rejecting the hypothesis of constant mean, when the statistic (2·3) exceeds d, has asymptotic power 1 for $S(.)$ such that $\Sigma\,\{S(t) - \bar{S}\}^2/\log T$ tends to infinity with T. This is the case under Assumption A·2 below.

4. THE EXAMPLE OF THE AMAZON RIVER

Daily stage, that is height, readings have been made since 1903 at Manaus, 18 km. up the Rio Negro estuary from the Amazon River in Brazil. In all 30 529 readings are available for analysis. Many developments have taken place in the Amazon basin this century, particularly a steady deforestation, so it seems of interest to examine this river stage series for monotonic trend; for descriptive analysis, see Sternberg (1987) and Brillinger (1988). Figure 2 shows the 8 year running mean level as defined by (2·4) with $V = 4 \times 365{\cdot}25$. This running mean can be quickly computed via a fast Fourier transform algorithm.

Fig. 2

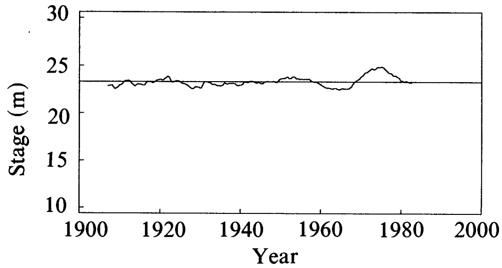

Fig. 3

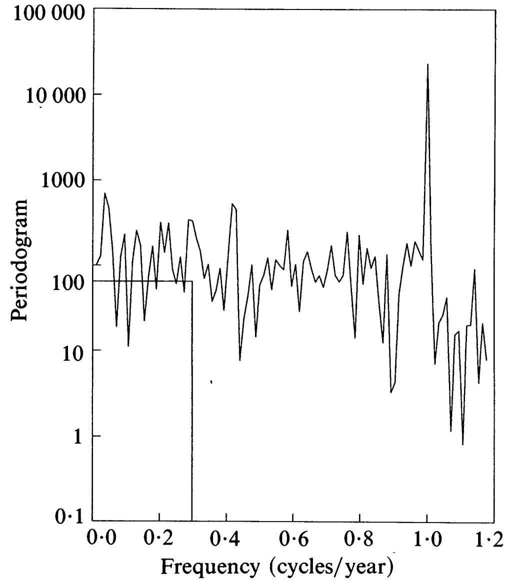

Fig. 2. Eight year running mean of daily data. Horizontal line at overall mean level.
Fig. 3. Low frequency portion of periodogram of data. Box in lower left-hand corner displays the estimated error spectrum at frequency 0 and bandwidth of estimate.

To study the low frequency character of the time series, Fig. 3 gives the periodogram at the lower frequencies. The high peak occurs at the seasonal frequency. This periodogram was computed via a fast Fourier transform of the $N = 30\,720$ observations obtained by adding 191 zeros to the end of the given data.

Turning to a formal examination of the data for a monotonic trend, the statistic (2·3) is evaluated. The spectrum at 0 is estimated by expression (2·7) with $L = 25$. For insight the value obtained and bandwidth employed have been indicated by a box added to Fig. 3. The value obtained for the statistic (2·3) is 1·961. The corresponding p-value for the question of monotonic increase is 0·025. By way of comparison, if a linear trend is fit to the series and the power spectrum at 0 estimated from the residuals of that fit, then the p-value obtained is 0·041. In summary there is some evidence for increasing trend or positive change in this data.

5. Discussion and concluding remarks

The noncentrality parameter occurring in expression (3·2) is

$$\sum c(t)S(t)/\{2\pi f_{EE}(0)\sum c(t)^2\}^{\frac{1}{2}}.$$

Under Assumption A·2 below, this is of order of magnitude $T/(\log T)^{\frac{1}{2}}$ and tends to infinity with T. If one employed the statistic $\hat{\beta}$ of (1·2) instead, the corresponding noncentrality parameter would be of order $T^{\frac{1}{2}}$ which tends to infinity more slowly. The corresponding test is therefore consistent, but has asymptotic efficiency 0 relative to the first.

It is clear that one could form other estimates of $S(t)$, such as the robust linear smoother of Cleveland (1979), or the monotonic smoother of Friedman & Tibshirani (1984). However the running mean is quickly computed and its statistical properties are simply derived.

It is often of substantial advantage to formulate statistical questions as ones of estimation rather than of testing. In the present circumstance one could estimate the increase $S(T-1)-S(0)$. The technique of Friedman & Tibshirani leads to an estimate of this quantity. To examine the hypothesis of no increase however one needs a corresponding uncertainty estimate. Such an estimate does not appear to be at present available for autocorrelated noise.

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APPENDIX 1

Mixing assumption and conditions on the trend

The cumulant functions of the stationary series $E(.)$ are defined by

$$c_{E\ldots E}(u_1, \ldots, u_{k-1}) = \text{cum}\,\{E(t+u_1), \ldots, E(t+u_{k-1}), E(t)\}, \tag{A·1}$$

for $k=2, 3, \ldots$, and the power spectrum at frequency λ by

$$f_{EE}(\lambda) = \frac{1}{2\pi} \sum_{u=-\infty}^{\infty} c_{EE}(u) \exp(-i\lambda u), \tag{A·2}$$

when this series converges.

Assumption A·1 (*Mixing*). For $k=2, 3, \ldots$

$$\sum_{u_1} \cdots \sum_{u_{k-1}} |c_{E\ldots E}(u_1, \ldots, u_{k-1})| < \infty, \tag{A·3}$$

with $u_1, \ldots, u_{k-1}$ running from $-\infty$ to ∞.

Assumption A·2. The signal has the form $S(t) = S^T(t) = g(t/T)$, with $g(.)$ square integrable, vanishing outside the unit interval, and where there is an α such that the integral modulus of continuity of $g(.)$ satisfies

$$\sup_{|v| \leq h} \int_0^1 |g(u+v) - g(u)|^2 \, du = O(h^\alpha), \tag{A·4}$$

as $h \to 0$.

Condition (A·4) allows steps in $g(.)$. In particular if $g(u)=0$ for $u \leq u_0$ and $=1$ for $u > u_0$, then $\alpha = 1$. In the case that $g(.)$ has bounded derivative on $[0, 1]$, $\alpha = 2$.

APPENDIX 2

Elementary results needed for proofs of theorems

LEMMA A·1. *For the coefficients $c(.)$ of (1·4), for fixed $u, u_1, \ldots, u_{k-1}$ and for $k=2, 3, \ldots$ one has*

$$\sum \{c(t)\}^2 \simeq \tfrac{1}{2} \log T, \tag{A·5}$$

$$\lim_{T\to\infty} \sum c(t+u)c(t)/\sum c(t)^2 = 1, \quad (A\cdot6)$$

$$\lim_{T\to\infty} \sum c(t+u_1)\ldots c(t+u_{k-1})c(t)/\{\sum c(t)^2\}^{k/2} = 0. \quad (A\cdot7)$$

The proof is by elementary analysis.

Next define

$$\tilde{S}(t) = \sum_{s=-V}^{V} S(t+s)/(2V+1), \quad (A\cdot8)$$

the running mean of the signal. One has the following.

LEMMA A·2. *Let $S(t)=g(t/T)$ with $g(.)$ satisfying Assumption* A·2. *Let $V=V^T$ be such that $V/T\to 0$ as $T\to\infty$. Then*

$$\sum_t |S(t)-\tilde{S}(t)|^2 = \frac{1}{T}\sum_j |s_j - \tilde{s}_j|^2 = O(V^\alpha T^{1-\alpha}). \quad (A\cdot9)$$

Proof. The first equality follows from Parseval's Theorem. Next, neglecting end effects as one may,

$$\sum |S(t)-\tilde{S}(t)|^2 \leqslant \frac{1}{2V+1}\sum_{|s|\leqslant V}\sum |S(t)-S(t+s)|^2 \leqslant \sup_{|s|\leqslant V}\sum |S(t)-S(t+s)|^2$$

$$\leqslant \sup_{|s|\leqslant V}\sum \left|g\left(\frac{t}{T}\right)-g\left(\frac{t+s}{T}\right)\right|^2 \simeq T \sup_{|v|\leqslant V/T}\int |g(u+v)-g(u)|^2\,du$$

and one has the indicated result. □

APPENDIX 3

Proof of asymptotic normality and consistency

THEOREM A·1. *Suppose that the series $E(.)$ satisfies Assumption* A·1. *Suppose that $c(.)$ is given by* (1·4). *Then*

$$\lim_{T\to\infty} \operatorname{var}\{\sum c(t)Y(t)\}/\sum c(t)^2 = 2\pi f_{EE}(0). \quad (A\cdot10)$$

Proof. From expression (2·2) the variance may be written

$$\sum_{u=-T+1}^{T-1} c_{EE}(u) \sum_{t=0}^{T-1-|u|} c(t+u)c(t).$$

The result now follows from (A·3) and the Dominated Convergence Theorem. □

THEOREM A·2. *Under the conditions of Theorem* A·1, *the variate $\Sigma\, c(t)Y(t)$ is asymptotically normal.*

Proof. The standardized cumulant of order k may be written

$$\textstyle\sum_{u_1}\ldots\sum_{u_{k-1}} c_{E\ldots E}(u_1,\ldots,u_{k-1})\sum_t c(t+u_1)\ldots c(t+u_{k-1})c(t)/\{\sum c(t)^2\}^{k/2}.$$

This tends to 0 for $k>2$ by (A·3), (A·7) and the Dominated Convergence Theorem. The asymptotic normality then follows from Lemma P4.5 of Brillinger (1975, p. 403). □

THEOREM A·3. *Let the series $E(.)$ satisfy Assumption* A·1 *and let the signal $S(.)$ satisfy Assumption* A·2. *Suppose that $L=L^T$ and $V=V^T$ tend to ∞ as $T\to\infty$ in such a way that L/T, V/T, $V^\alpha T^{1-\alpha}/L\to 0$. Then $\hat{f}_{EE}(0)$ is a consistent estimate of $f_{EE}(0)$.*

Proof. Since $Y(t) = S(t) + E(t)$ one has $y_j = s_j + \varepsilon_j$ and

$$|\hat{\varepsilon}_j|^2 = |s_j - \tilde{s}_j|^2 + \{\text{conjg}\,(s_j - \tilde{s}_j)\}(\varepsilon_j - \tilde{\varepsilon}_j) + (s_j - \tilde{s}_j)\{\text{conjg}\,(\varepsilon_j - \tilde{\varepsilon}_j)\} + |\varepsilon_j - \tilde{\varepsilon}_j|^2.$$

To begin, one notes, following (A·9), that the first term here has a negligible effect on (2·7). Next one requires that

$$\sum \frac{1}{2\pi T} |\varepsilon_j - \tilde{\varepsilon}_j|^2 \Big/ \sum (1 - a_j)^2,$$

is a consistent estimate of $f_{EE}(0)$. This follows from classical arguments for evaluating the mean and variance of quadratic spectral estimates; see Grenander & Rosenblatt (1957) and Parzen (1957) for example.

Finally, from the previous two results, the cross-product terms may be neglected. □

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