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Jump and sharp cusp detection by wavelets

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SUMMARY

A method is proposed to detect jumps and sharp cusps in a function which is observed with noise, by checking if the wavelet transformation of the data has significantly large absolute values across fine scale levels. Asymptotic theory is established and practical implementation is discussed. The method is tested on simulated examples, and applied to stock market return data.

Some key words: Convergence rate; Estimation; Hypothesis; Jump; Nonparametric regression; Sharp cusp; Wavelet transformation; White noise; White noise model.

1. INTRODUCTION

The analysis of change-points, which describe sudden localised changes, has recently found increasing interest. Change-points can be used to model practical problems arising in fields such as quality control, economics, medicine, signal and image processing, and physical sciences. For example, in electroencephalogram signals, sharp cusps exhibit the accelerations and decelerations in the beating of the hearts. Many practical problems like this involve functions which have jumps and sharp cusps.

The recently developed theory of wavelets has drawn much attention from both mathematicians, statisticians and engineers. In the seminal work of Donoho (1993), Donoho & Johnstone (1994, 1995a,b) and Donoho, Johnstone et al. (1995), orthonormal bases of compactly supported wavelets have been used to estimate functions. The theory of wavelets permits decomposition of functions into localised oscillating components. This is an ideal tool to study localised changes such as jumps and sharp cusps in one dimension as well as several dimensions. Unlike traditional smoothing methods based on a fixed spatial scale, the wavelet method is a multiresolution approach and has local adaptivity. In this paper we consider only jump and sharp cusp detection in one dimension.

There is a great amount of statistical literature on change-points (Basseville, 1988; Basseville & Nikiforov, 1993). Wahba (1984) and Engle et al. (1986) were the first to estimate curves with discontinuities in derivatives, assuming the locations of the jumps are known. McDonald & Owen (1986) proposed an algorithm to compute estimates of regression functions when discontinuities are present. Yin (1988) used one-sided moving averages to find the locations of jumps in a function. Lombard (1988) described jump detection by Fourier analysis. Cline & Hart (1991) considered detecting jumps in derivatives. Müller (1992) estimated the location of a jump and its jump size by boundary kernels. Eubank & Speckman (1995) used a semiparametric approach to detect the discontinuities in derivatives of regression functions. Hall & Titterington (1992) studied edge-preserving and peak-preserving by smoothing. Grossmann (1986) and Mallat & Hwang (1992) used wavelet transformation to detect singularities and edges in computer images.

The paper is organised as follows. Sections 2 and 3 introduce the white noise model and wavelet transformation, respectively. Testing hypotheses and estimation are considered in §§ 4 and 5. Section 6 discusses implementation of the detection in practice. Simulation results and an application to a real example are reported in this section. Concluding remarks are given in § 7. Proofs are collected in the Appendix.

2. The white noise model

We say a function f has an α-cusp at x_0 if there exists a positive constant K such that, as h tends to zero from left or right,

$$|f(x_0+h)-f(x_0)| \geqslant K|h|^{\alpha}. \tag{1}$$

For the case $\alpha=0$, f has a jump at x_0. This paper considers sharp cusp detection, so from now on we restrict to the case $0 \leqslant \alpha < 1$.

Suppose f is observed from the white noise model

$$Y(dx)=f(x)\,dx+\tau W(dx), \quad x \in [0, 1], \tag{2}$$

where W is a standard Wiener process, τ is a formal noise level parameter which we think of as small, and f is an unknown function which may have jumps and sharp cusps. The problem is to detect these jumps and cusps.

The white noise model (2) is a generalisation of the diffusion model used to study change-points by Pollak & Siegmund (1985). It is also closely related to the following nonparametric regression model:

$$y_i=f(x_i)+\sigma z_i \quad (i=1, \ldots, n), \tag{3}$$

with $x_i=i/n$, the (z_i) independent standard normal errors, $\sigma>0$, and f an unknown function. Define the regression process $\{Y_n(x): x \in [0, 1]\}$ via $x_0=0$, $Y_n(0)=0$ and $Y_n(x_i)=y_1+\ldots+y_i$ for $i=1, \ldots, n$, with interpolation by a Wiener process W for $x_i \leqslant x < x_{i+1}$. Then Y_n is a white noise process with the function $f_n(x)=f(x_i)$ for $x_i \leqslant x < x_{i+1}$ and $\tau=\sigma n^{-\frac{1}{2}}$ (Donoho & Johnstone, 1995a). We will use this relationship to compare rates of convergence for the models (2) and (3).

3. Wavelet transformation

Let ψ be a Daubechies wavelet (Chui, 1992, Ch. 1, 7; Daubechies, 1992, Ch. 1, 6; Donoho & Johnstone, 1995a), and define $\psi_s(x)=s^{-\frac{1}{2}}\psi(x/s)$. The wavelet transformation of f is defined as $Tf(s, x)=\int \psi_s(x-u)f(u)\,du$. The wavelet transformation $Tf(s, x)$ is a function of the scale, or frequency, s and the spatial position, or time, x. The plane defined by the pair of variables (s, x) is called the scale-space, or time-frequency, plane (Chui, 1992, Ch. 1, 3; Daubechies, 1992, Ch. 2, 3; Mallat & Hwang, 1992).

For a compactly supported wavelet, the value of $Tf(s, x)$ depends upon the value of f in a neighbourhood of x of size proportional to the scale s. At small scales, $Tf(s, x)$ provides localised information such as local regularity on $f(x)$. The local regularity of a function is often measured with Lipschitz exponents as follows.

A function $f(x)$ is said to be Lipschitz α at x_0 if there exists a positive constant K such that, as h tends to zero,

$$|f(x_0+h)-f(x_0)| \leqslant K|h|^{\alpha}.$$

The function $f(x)$ is said to be uniformly Lipschitz α over $[0, 1]$ if it is Lipschitz α at all points in $(0, 1)$.

Mathematically the global and local Lipschitz regularity can be characterised by the asymptotic decay of wavelet transformation at small scales (Daubechies, 1992, Theorems 2.9.1–2.9.4, pp. 45–9). For example, if f is differentiable at x, $Tf(s, x)$ has the order $s^{3/2}$ as s tends to zero, and if f has an α-cusp at x, the maximum of $Tf(s, x)$ over a neighbourhood of x of size proportional to the scale s converges to zero at a rate no faster than $s^{\alpha+\frac{1}{2}}$ as s tends to zero.

The wavelet transformation of the white noise $W(dx)$ is defined to be $TW(s, x) = \int \psi_s(x-u)W(du)$. The wavelet transformation of Y is

$$TY(s, x) = \int \psi_s(x-u)Y(du) = Tf(s, x) + \tau TW(s, x). \tag{4}$$

At a given scale s, $TW(s, x)$ is a stationary Gaussian process with zero mean and covariance function

$$E\{TW(s, x)TW(s, y)\} = \int \psi_s(x-u)\psi_s(y-u)\, du, \tag{5}$$

$$\operatorname{var}\{TW(s, x)\} = \int \{\psi(u)\}^2\, du = 1.$$

Note that $TW(s, x)$ follows a standard normal distribution and that the orders of $Tf(s, x)$ are, respectively, $s^{\alpha+\frac{1}{2}}$ and $s^{3/2}$ for the two cases that $f(x)$ has an α-cusp at x and $f(x)$ is differentiable at x. By (4) we can see that, at a very fine scale s, $TY(s, x)$ is dominated by $\tau TW(s, x)$, while, at a coarse scale s, $Tf(s, x)$ dominates $TY(s, x)$. The localised information of $f(x)$ is provided by $Tf(s, x)$ at fine scales, so the wavelet transformation at finer scales can detect local changes more precisely. Our idea is to select fine scales s_τ such that, at those x where $f(x)$ is differentiable, the orders of $Tf(s_\tau, x)$ and $\tau TW(s_\tau, x)$ are balanced. If f has sharp cusps, for nearby x, $TY(s_\tau, x)$ will be dominated by $Tf(s_\tau, x)$ and hence significantly larger than the others. Therefore, the sharp cusps will be detected by checking the values of $TY(s_\tau, x)$.

Throughout this paper, take η to be any constant greater than 1, and let s_τ be of exact order $(\tau^2|\log \tau|^\eta)^{1/(2\alpha+1)}$ as $\tau \to 0$. Denote by $\operatorname{supp}(\psi)$ the support of ψ.

4. Testing hypotheses

Consider the testing problem H_0: f is differentiable, against H_1: f has α_i-cusps, $i = 1, \dots, q$ $(q \geqslant 1)$, where $\alpha_i \leqslant \alpha < 1$.

Under H_0, f is a smooth function, and hence $TY(s_\tau, x)$ is dominated by the wavelet transformation of Brownian motion, which is a stationary Gaussian process. However, under H_1, f has jumps and/or sharp cusps, so for nearby x, $TY(s_\tau, x)$ is dominated by $Tf(s_\tau, x)$, whose absolute value is large. Therefore, the maximum of $|TY(s_\tau, x)|$ over $0 \leqslant x \leqslant 1$ is of much larger order under H_1 than under H_0, and thus can serve as a test statistic.

The following theorem gives an approximate critical value $C_{\tau,\gamma}$ for this test, for size γ.

THEOREM 1. *If* $0<\gamma<1$ *then, under* H_0,

$$\lim_{\tau\to 0} \text{pr}\left\{\max_{0\leqslant x\leqslant 1} |TY(s_\tau, x)| \geqslant C_{\tau,\gamma}\right\} = \gamma,$$

where

$$C_{\tau,\gamma} = \tau(2|\log s_\tau|)^{-\frac{1}{2}}\left\{2|\log s_\tau| + \log\left(\left[\int \{\psi'(u)\}^2\, du\right]^{\frac{1}{2}} \Big/ (2\pi)\right) - \log\{-\log(1-\gamma)/2\}\right\}. \tag{6}$$

5. ESTIMATION

5·1. *One jump or sharp cusp*

Suppose f has an α-cusp at θ and is differentiable elsewhere. An estimate of θ is

$$\hat\theta = \arg\max_{0\leqslant x\leqslant 1} \{|TY(s_\tau, x)|\}. \tag{7}$$

THEOREM 2. *We have*

$$\lim_{\tau\to 0} \text{pr}\{s_\tau^{-1}(\hat\theta-\theta) \in \text{supp}\,(\psi)\} = 1.$$

The compact support of ψ *implies that the estimate* $\hat\theta$ *has the convergence rate* s_τ.
Moreover, suppose that

$$f(x) = f(\theta) + A_1|x-\theta|^\alpha + o(|x-\theta|^\alpha)$$

as $x\to\theta$ *if* $\alpha>0$, *and* $f(\theta+)-f(\theta-) = A_2$ *if* $\alpha=0$, *where* $A_i \neq 0$ $(i=1,2)$. *Then, as* $\tau\to 0$, $(\hat\theta-\theta)/s_\tau$ *converges in probability to the location of the maximum of*

$$\left\{\left|\int \psi(u-t)\,|u|^\alpha\, du\right| : t\in \text{supp}\,(\psi)\right\}$$

if $\alpha>0$, *or of* $\{|\int \psi(u-t)\,\text{sign}\,(u)\,du| : t\in\text{supp}\,(\psi)\}$ *if* $\alpha=0$.

Since jump detection has been studied in the nonparametric regression setting, we compare the above convergence rate with those in the literature. For the jump case, $\alpha=0$, by Theorem 2 the convergence rate of $\hat\theta$ is $\tau^2|\log\tau|^\eta$ for the white noise model. Using the relation between the models (2) and (3) described in § 2 and letting $\tau=\sigma n^{-\frac{1}{2}}$, we obtain that this rate corresponds to the rate $n^{-1}(\log n)^\eta$ for the nonparametric regression model, which is known to be the best possible convergence rates in the literature (Müller, 1992). Hence, the estimate $\hat\theta$ achieves the optimal convergence rate.

Since, for $0\leqslant\alpha\leqslant 1$, $\frac{1}{3}\leqslant 1/(2\alpha+1)\leqslant 1$, from Theorem 2 and the relation between the models (2) and (3) we have that, without knowing the sharpness of the cusp, the convergence rates are at least $\tau^{2/3}|\log\tau|^{\eta/3}$ for the white noise model and $n^{-1/3}(\log n)^{\eta/3}$ for the nonparametric regression model. Such rates are higher than that of the mode estimate when the function is twice differentiable near the mode (Eddy, 1980, 1982). This confirms that it is easier to detect a spiked bump than a smooth one.

5·2. *Several jumps and/or sharp cusps*

Suppose f has q cusps with an α_i-cusp at θ_i $(i=1,\ldots,q)$, where q is a finite integer, $\alpha_1\leqslant\ldots\leqslant\alpha_q$. Suppose also that f is differentiable at all points except for $\theta_1,\ldots,\theta_q$.

Assume that $\alpha_1 \leqslant \ldots \leqslant \alpha_q \leqslant \alpha < 1$ and α is known. First suppose q is known. The estimate $(\hat{\theta}_1, \ldots, \hat{\theta}_q)$ of $(\theta_1, \ldots, \theta_q)$ is constructed as follows:

1. Find the location, $\hat{\theta}_1$, of the maximum of $|TY(s_\tau, x)|$ over $[0, 1]$.
2. Find the location, $\hat{\theta}_2$, of the maximum of $|TY(s_\tau, x)|$ over
$$[0, 1]\backslash\{\hat{\theta}_1 + s_\tau x : x \in \operatorname{supp}(\psi)\}.$$
3. Continue the procedure until the location, $\hat{\theta}_q$, of the maximum of $|TY(s_\tau, x)|$ over
$$[0, 1] \Big\backslash \bigcup_{i=1}^{q-1} \{\hat{\theta}_i + s_\tau x : x \in \operatorname{supp}(\psi)\}$$
is found.

The following theorem establishes convergence rates for the above method.

THEOREM 3. *We have*
$$\sum_{i=1}^{q} (\hat{\theta}_i - \theta_i)^2 = O_p(s_\tau^2).$$

Now consider the case that q is unknown. Theorem 1 implies that, with probability tending to $1-\gamma$, $|TY(s_\tau, x)| \leqslant C_{\tau,\gamma}$ at those x at which f has no jump or sharp cusp. So we take $C_{\tau,\gamma}$ as a threshold and use the procedure described by steps 1–3 above to find all the local maxima which exceed $C_{\tau,\gamma}$. Let $\hat{q}$ be the number of those maxima and $\hat{\theta}_1, \ldots, \hat{\theta}_{\hat{q}}$ their locations. Then we simultaneously estimate q and $\theta_1, \ldots, \theta_q$ by $\hat{q}$ and $\hat{\theta}_1, \ldots, \hat{\theta}_{\hat{q}}$.

THEOREM 4. *As* $\tau \to 0$ *and then* $\gamma \to 0$,
$$\operatorname{pr}(\hat{q} = q) \to 1, \quad \sum_{i=1}^{\hat{q}} (\hat{\theta}_i - \theta_i)^2 = O_p(s_\tau^2).$$

Theorem 4 shows that, when the number of jumps and sharp cusps is unknown, the above method will asymptotically pick up all the jumps and sharp cusps, and will estimate their locations with the same convergence rate as in Theorem 3 for the case that the number of jumps and sharp cusps is known.

6. IMPLEMENTATION IN PRACTICE

6·1. *Discrete observations*

In practice, we might observe $Y(x)$ only at x discrete values $x = i/n$ $(i = 1, \ldots, n = 2^J)$. Or equivalently, we observe f from the model (3); that is $y_i = f(i/n) + \sigma z_i$, $\sigma > 0$, $z_i \sim N(0, 1)$ $(i = 1, \ldots, n = 2^J)$. Consequently a discrete version of the wavelet transformation must be performed.

6·2. *Discrete wavelet transformation*

The discrete wavelet transformation is a discretised version of the continuous wavelet transformation and can be written as linear transformation involving a $n \times n$ orthogonal matrix $\mathcal{W}$ which depends on the wavelet and the boundary adjustment (Cohen et al., 1993; Daubechies, 1994). Let $y = (y_1, \ldots, y_n)$. The discrete wavelet transformation of the data y is given by $w = \mathcal{W}y$. Because $\mathcal{W}$ is orthogonal, the inverse discrete wavelet transformation is $y = \mathcal{W}^{\mathrm{T}} w$. The $n-1$ elements of w are indexed dyadically: $w_{j,k}$ for

$k=0,\ldots,2^j-1$, $j=0,\ldots,J-1$, and the remaining element is labelled $w_{-1,0}$. The quantity $w_{j,k}$ is called the empirical wavelet coefficient at level j and position $k2^{-j}$ for $k=0,\ldots,2^j-1$, $j=0,\ldots,J-1$.

The rows of $\mathscr{W}$ correspond to a discretised version of the wavelets ψ_λ, and $w_{j,k}$ relates to a discretised version of the continuous wavelet transformation $TY(2^{-j},k2^{-j})$ for $k=0,\ldots,2^j-1$, $j=0,\ldots,J-1$. Indeed, if we dyadically index the first $n-1$ rows of $\mathscr{W}$ by (2^j+k) for $k=0,\ldots,2^j-1$, $j=0,\ldots,J-1$, and denote by $W_{jk}(i)$ the ith element of the (2^j+k)th row of $\mathscr{W}$, then $n^{\frac{1}{2}}W_{jk}(i)$ is approximately equal to $2^{j/2}\psi(2^jx)$ for $x=i/n-k2^{-j}$, and hence $n^{\frac{1}{2}}w_{j,k}$ approximates $TY(2^{-j},k2^{-j})$ (Donoho & Johnstone, 1994).

Mallat's pyramidal algorithm (Mallat, 1989; Chui, 1992, pp. 20–1) requires only $O(n)$ operations for computing the discrete wavelet transformation and reconstruction of the discrete wavelet transformation.

6·3. *Threshold selection*

Since discretely spaced data are observed, we take $\tau=\sigma n^{-\frac{1}{2}}$ in (6) and obtain a threshold C_n which is used to select some number of jumps and sharp cusps. Simple algebra shows that

$$C_n=\sigma(2\log n/n)^{\frac{1}{2}}\{1+O(1/\log n)\}.$$

The leading term in C_n is equal to the universal threshold of Donoho & Johnstone (1994). Following Donoho & Johnstone (1994), we estimate σ by the median absolute deviation of the wavelet coefficients at the finest level $J-1$, divided by 0·6745.

6·4. *Simulations*

To test the method, two simulated examples are carried out and illustrated in Figs 1 and 2. For the two examples, $n=2^{10}$. The underlying function in Example 1 has a jump and an unbalanced cusp. Example 2 is a modified example of Donoho & Johnstone (1994) and the true function has a jump, a sharp cusp and some smooth bumps.

Let Daubechies DN wavelet denote Daubechies' compactly supported wavelet with maximum number of vanishing moments for the support width, and with the extremal phase choice, for integer parameter N (Daubechies, 1992, Ch. 6). For example, the Daubechies D1 wavelet is the Haar wavelet.

By Mallat's pyramid algorithm, we use the Daubechies D1, D2 or D3 wavelets with reflecting boundary condition to compute the empirical wavelet coefficients at level j and position $k2^{-j}$ for $k=0,\ldots,2^j-1$, $j=0,\ldots,9$. The boundary condition affects only wavelet coefficients near the two boundary points and has little impact on detecting jumps and sharp cusps away from the boundary points.

By checking the empirical wavelet coefficients at these 10 levels, we can find dyadic intervals at some levels whose corresponding absolute empirical wavelet coefficients exceed the threshold line and are significantly larger than the others. Thus cusps must be located in the dyadic intervals, including the endpoints. Since at high resolution levels the dyadic intervals are very narrow, with widths proportional to the scale $s=2^{-j}$, the cusps will be located with sufficient accuracy. In Figs 1 and 2, (a), (b) and (c) are, respectively, the true curves, noisy curves and absolute empirical wavelet coefficients at levels $j=6$ or 7. The horizontal lines in Figs 1(c) and 2(c) are the threshold lines. From the Figs 1(c) and 2(c), we can see that the empirical wavelet coefficients are significantly large and exceed the threshold lines only at the locations where the functions have jumps and sharp cusps.

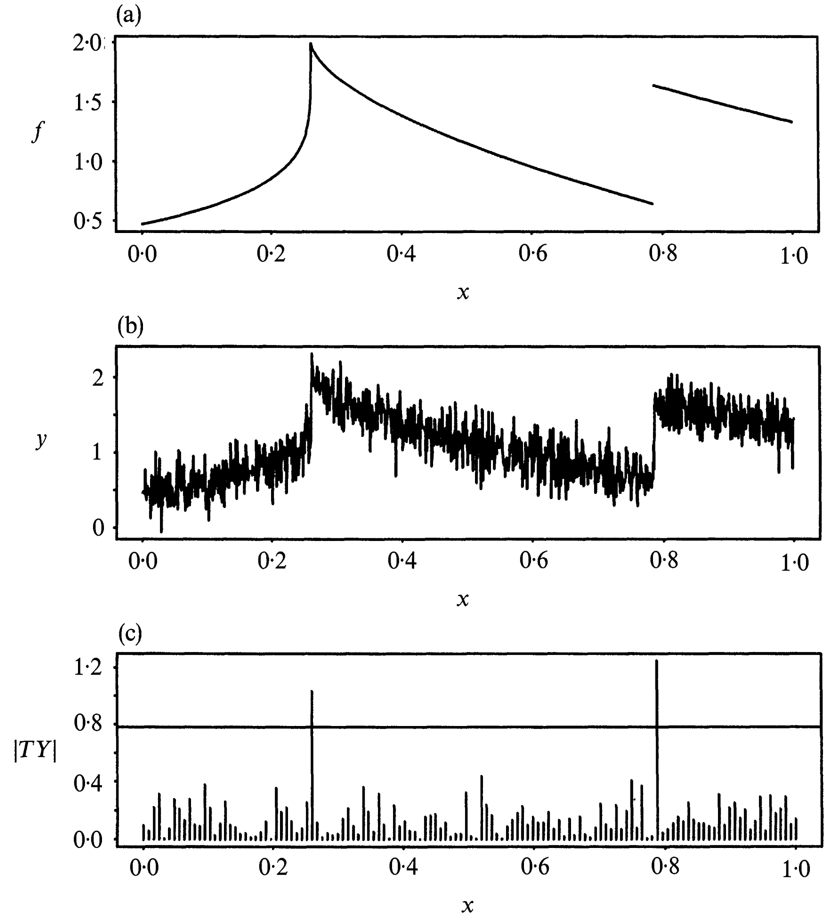

Fig. 1. Data simulated from model $y_i = f(i/n) + \varepsilon_i$, $f(x) = 2 - 2|x - 0{\cdot}26|^{1/5} 1(x \leqslant 0{\cdot}26) - 2|x - 0{\cdot}26|^{3/5} 1(x > 0{\cdot}26) + 1(x \geqslant 0{\cdot}78)$, $\varepsilon_i \sim N(0, \sigma^2)$, $\sigma = 0{\cdot}2$, $n = 1024$; (a) true curve, (b) noisy curve and (c) absolute empirical wavelet coefficients, at level $j = 7$.

At the suggestion of a referee, we decreased the signal-to-noise ratio in the simulated examples to check the performance of the detection. With $n = 1024$ in Example 1, we increased σ from 0·2 to 1·0 in steps of 0·2. The detection works well for σ up to 0·4. After that, the jump and the cusp become harder and harder to detect. They tend to be detected by wavelet coefficients at lower and lower resolution levels and thus the detection is less and less precise. In particular, the cusp is often either located in low resolution levels or goes undetected. Figure 3 is the plot of the case with $\sigma = 0{\cdot}6$. The jump is located at level 5 while the cusp is not detected at levels higher than 3. From Fig. 3(b) we can see that the cusp is totally buried in the noise. For $\sigma \geqslant 1$ the method completely breaks down and the jump and the cusp are invisible in the plot of noisy data.

6·5. *A real example*

Now we consider the stock market return data in the United States. Figure 4(a) is the plot of the 468 monthly data from 1953 to 1991. It is well known in financial economics that stock market return data can be modelled by the white noise model (Huang & Litzenberger, 1988).

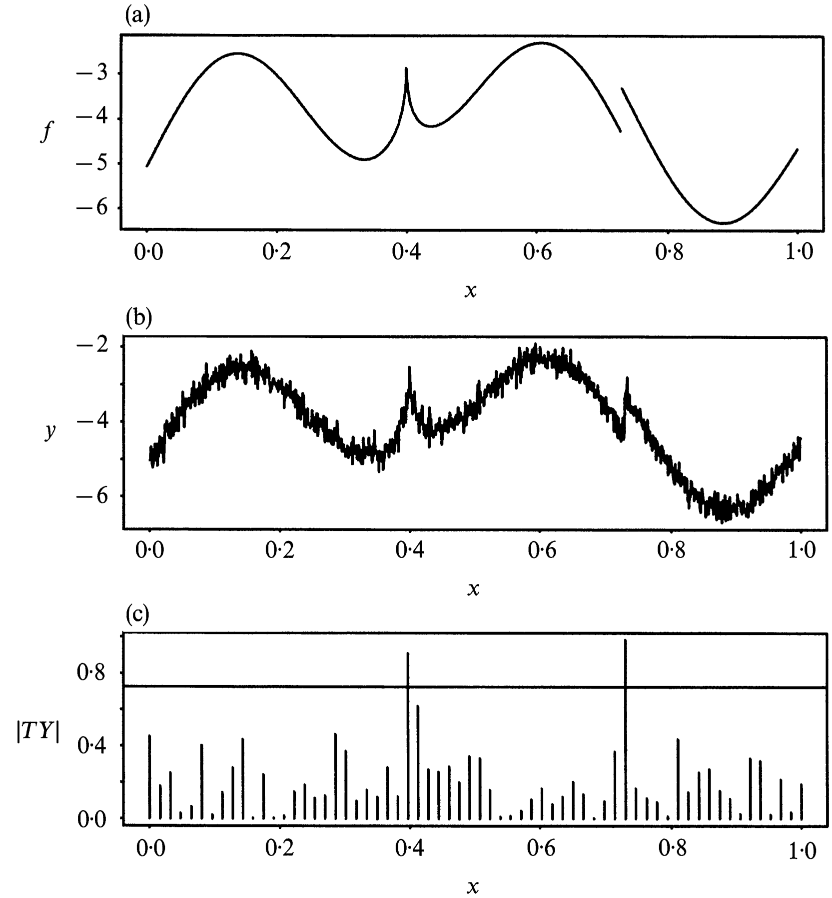

Fig. 2. Data simulated from model $y_i = f(i/n) + \varepsilon_i$, $f(x) = 2\sin(4\pi x) - 6|x - 0{\cdot}4|^{3/10} - 0{\cdot}5\,\mathrm{sign}\,(0{\cdot}7 - x)$, $\varepsilon_i \sim N(0, \sigma^2)$, $\sigma = 0{\cdot}2$, $n = 1024$; (a) true curve, (b) noisy curve and (c) absolute empirical wavelet coefficients, at level $j = 6$.

In order to implement Mallat's pyramid algorithm, 44 data points should be added to make the total number a power of 2. Since the data at the beginning are almost equal to zero, we add 44 zeros before the beginning of the data. These added data affect only the wavelet coefficients near the boundary points, so they will have little impact on detecting jumps and sharp cusps.

The Daubechies D8 wavelet with reflecting boundary condition is used to compute the empirical wavelet coefficients. Figures 4(b) and (c) are the plots of the empirical wavelet coefficients at levels 2^{-6} and 2^{-7}, respectively. The horizontal lines are the threshold lines.

In Figs 4(b) and (c), the empirical wavelet coefficients exceed the threshold lines at the two locations. The locations of these large empirical wavelet coefficients are near the observations 262 (October 1974) and 418 (October 1987), so there are local structural changes near the corresponding times. They are caused by the recession in 1974 and the New York stock market crash in 1987.

7. Concluding remark

Detection by wavelets is a multiresolution technique. We check empirical wavelet coefficients across resolution levels and locate jumps and sharp cusps by empirical wavelet

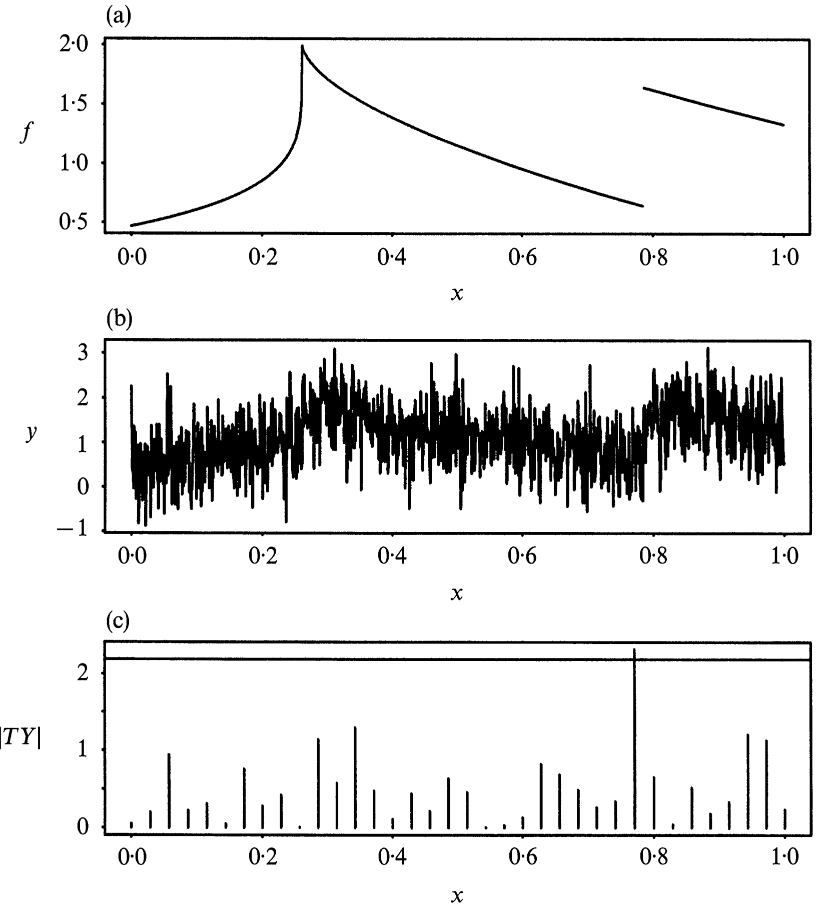

Fig. 3. Data simulated from model $y_i = f(i/n) + \varepsilon_i$, $f(x) = 2 - 2|x - 0{\cdot}26|^{1/5} 1(x \leqslant 0{\cdot}26) - 2|x - 0{\cdot}26|^{3/5} 1(x > 0{\cdot}26) + 1(x \geqslant 0{\cdot}78)$, $\varepsilon_i \sim N(0, \sigma^2)$, $\sigma = 0{\cdot}6$, $n = 1024$; (a) true curve, (b) noisy curve and (c) absolute empirical wavelet coefficients, at level $j = 5$.

coefficients at these levels. The multiresolution approach has local adaptivity and hence has advantages over existing smoothing methods based on fixed spatial scale, such as Fourier series methods and fixed bandwidth kernel methods. See also Donoho & Johnstone (1994). For example, because of different local features, a jump should be detected more easily and more accurately than a cusp. Detection by wavelets can locate the jump more precisely by wavelet coefficients at higher resolution levels and at the same time detect the cusp by wavelet coefficients at lower resolution levels. The method has been employed in construction of the wavelet estimate of a function with jumps to reduce Gibbs errors (Wang, 1995).

The paper leaves some open practical issues such as adaptive selection of wavelets as well as scale parameters. However, practical problems can really be solved by the approach employed in §§ 6·4 and 6·5: comparing wavelet coefficients at all levels with the threshold, we select large wavelet coefficients and then use these to detect jumps and sharp cusps. Theoretically, if we have continuous observations, the type of wavelet does not affect the detection. In practice, however, only discretely spaced data are available, so the wavelet used often has some effects on levels at which the jumps and sharp cusps are detected and

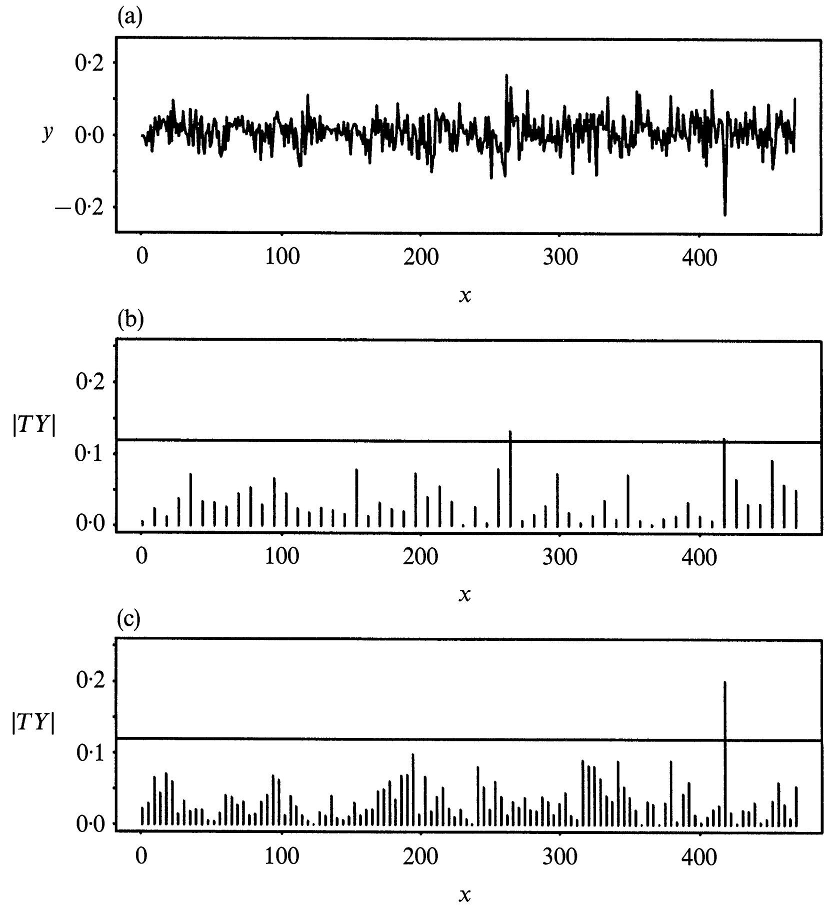

Fig. 4. Monthly stock market return data in the United States from 1953 to 1991: (a) the plot of 468 data; (b) and (c) are the empirical wavelet coefficients at levels $j = 6$ and $j = 7$, respectively.

hence the accuracy of the detection. We can try several wavelets, say Daubechies D1–D10 wavelets, and select that for which the jumps and sharp cusps can be detected at the highest possible levels. This was done for the examples presented in § 6.

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Appendix

Proofs

We shall need the following result.

LEMMA 1. *If ψ is differentiable, then, with probability one, there exists a positive constant $K < \infty$*

such that, for all x and small s,

$$|TW(s, x)| \leqslant K|\log s|^{\frac{1}{2}}.$$

Proof. Note that, for Brownian motion, with probability one,

$$\lim_{h\to 0}\left\{(h|\log h|)^{-\frac{1}{2}} \max_{0\leqslant s\leqslant 1-h} \max_{0\leqslant t\leqslant h} |W(s+t)-W(t)|\right\} = 1.$$

Then

$$|TW(s, x)| = s^{-\frac{1}{2}}\left|\int \psi(v)\, dW_{x+sv}\right|$$

$$= s^{-\frac{1}{2}}\left|\int \psi'(v) W_{x+sv}\, dv\right|$$

$$= s^{-\frac{1}{2}}\left|\int \psi'(v)(W_{x+sv} - W_x)\, dv\right|$$

$$\leqslant |\log s|^{\frac{1}{2}} \int |\psi'(v)|\,|v|^{\frac{1}{2}}\, dv + \int |\psi'(v)|\,|v \log v|^{\frac{1}{2}}\, dv. \qquad \square$$

Proof of Theorem 1. Under H_0, f is a smooth function, so by Theorem 2.9.1 of Daubechies (1992, p. 45) we have

$$|Tf(s_\tau, x)| \leqslant K s_\tau^{3/2},$$

$$TY(s, x) = \tau TW(s_\tau, x) + O(s_\tau^{3/2}) = \tau TW(s_\tau, x)\{1 + o_p(1)\}.$$

Note that $TW(s_\tau, x)$ is a stationary Gaussian process, and, by (5), its covariance function satisfies

$$E\{TW(s, x)TW(s, y)\} = 1 - \frac{1}{2}\int \{\psi'(u)\}^2\, du\, (x-y)^2/s^2 + o\{(x-y)^2/s^2\}.$$

Then Theorem 1 follows from Corollary A1 of Bickel & Rosenblatt (1973). See also the proof of Theorem 3.1 of Bickel & Rosenblatt (1973). $\square$

Proofs of Theorems 2 *and* 3. Since the proofs are similar, we give only the argument to prove Theorem 2.

Let K be a generic constant whose value may change from line to line. Since f is differentiable at all points except θ, by Theorem 2.9.1 of Daubechies (1992, p. 45), we obtain that, for all (s_τ, x) with $(\theta - x)/s_\tau \notin \text{supp}\,(\psi)$,

$$|Tf(s_\tau, x)| \leqslant K s_\tau^{3/2}.$$

Note that $f(x)$ has an α-cusp at θ. By Theorems 2.9.3 and 2.9.4 of Daubechies (1992, p. 49), we have

$$\max\{|Tf(s_\tau, x)| : (\theta - x)/s_\tau \in \text{supp}\,(\psi)\} \geqslant K s_\tau^{\alpha+\frac{1}{2}},$$

since otherwise, f will be 'at least' Lipschitz α at θ. The above two inequalities together with Lemma 1 imply that

$$\begin{aligned}\max\{|TY(s_\tau, x)| : (\theta - x)/s_\tau \in \text{supp}\,(\psi)\} &\geqslant K(s_\tau^{\alpha+\frac{1}{2}} - \tau|\log s_\tau|^{\frac{1}{2}})\\ &\geqslant K\tau(|\log \tau|^{\eta/2} - |\log \tau|^{\frac{1}{2}})\\ &\geqslant K\tau|\log \tau|^{\eta/2}(1 - |\log \tau|^{(1-\eta)/2}), \qquad \text{(A1)}\end{aligned}$$

and, for all (s_τ, x) with $(\theta - x)/s_\tau \notin \text{supp}\,(\psi)$,

$$|TY(s_\tau, x)| \leqslant K(s_\tau^{3/2} + \tau|\log s_\tau|^{1/2}) \leqslant K\tau|\log \tau|^{1/2}. \qquad \text{(A2)}$$

Note that $\eta > 1$. Thinking of $|TY(s_\tau, x)|$ as a function of x, by (A1) and (A2) we can easily see that, as $\tau \to 0$, with probability tending to one, the maximum of $|TY(s_\tau, x)|$ will be achieved at some $\theta + s_\tau t$, where $t \in \text{supp}(\psi)$. The first part of Theorem 2 is proved.

Now we show the second part. We have shown that, with probability tending to one, $(\hat\theta - \theta)/s_\tau \in \text{supp}(\psi)$. Thus the limit of $s_\tau^{-1}(\hat\theta - \theta)$ is determined by the asymptotic behaviour of $\{TY(s_\tau, \theta + s_\tau t), t \in \text{supp}(\psi)\}$. The following computations use the fact that $\int \psi(u)\,du = 0$. For $t \in \text{supp}(\psi)$, if $\alpha > 0$,

$$\begin{aligned} s_\tau^{-\frac{1}{2}} TY(s_\tau, \theta + s_\tau t) &= \int \psi(u) f(\theta + s_\tau u + s_\tau t)\,du + \tau s_\tau^{-1} \int \psi(u)\,dW_{\theta + s_\tau(u+t)} \\ &= A_1 s_\tau^\alpha \int \psi(u)|u+t|^\alpha\,du + \tau s_\tau^{-\frac{1}{2}} \int \psi(u)\,d\tilde W_{u+t} + o(s_\tau^\alpha) \\ &= A_1 s_\tau^\alpha \left\{ \int \psi(u-t)|u|^\alpha\,du + o_p(1) \right\}, \end{aligned} \qquad \text{(A3)}$$

and if $\alpha = 0$,

$$\begin{aligned} s_\tau^{-\frac{1}{2}} TY(s_\tau, \theta + s_\tau t) &= \int \psi(u) f(\theta + s_\tau u + s_\tau t)\,du + \tau s_\tau^{-1} \int \psi(u)\,dW_{\theta + s_\tau(u+t)} \\ &= \int \psi(u)\{f(\theta-)1(u+t<0) + f(\theta+)1(u+t \geqslant 0)\}\,du \\ &\quad + \tau s_\tau^{-\frac{1}{2}} \int \psi(u)\,d\tilde W_{u+t} + o(1) \\ &= 2^{-1}\{f(\theta+) - f(\theta-)\} \int \psi(u)\,\text{sign}(u+t)\,du + \tau s_\tau^{-\frac{1}{2}} \int \psi(u)\,d\tilde W_{u+t} + o(1) \\ &= 2^{-1} A_2 \left\{ \int \psi(u-t)\,\text{sign}(u)\,du + o_p(1) \right\}, \end{aligned} \qquad \text{(A4)}$$

where $\tilde W$ is a standard two-sided Brownian motion. Since $|TY(s_\tau, x)|$ achieves its maximum at $\hat\theta$, (A3) and (A4) imply that, as $\tau \to 0$, $s_\tau^{-1}(\hat\theta - \theta)$ weakly converges to the location of the maximum of the function

$$\left\{ \left| \int \psi(u-t) \right| u|^\alpha\,du| : t \in \text{supp}(\psi) \right\} \quad (\alpha > 0),$$

or of

$$\left\{ \left| \int \psi(u-t)\,\text{sign}(u)\,du \right| : t \in \text{supp}(\psi) \right\} \quad (\alpha = 0). \qquad \square$$

Proof of Theorem 4. Theorem 1 implies that, with probability tending to $1-\gamma$, $|TY(s_\tau, x)| \leqslant C_{\tau,\gamma}$ for those x at which f has no jump or sharp cusp. Using the arguments in the proof of Theorem 2 we can show that, as $\tau \to 0$, with probability tending to $1-\gamma$, $|TY(s_\tau, x)|$ will exceed $C_{\tau,\gamma}$ at only those x such that $(x - \theta_i)/s_\tau \in \text{supp}(\psi)$ for some $i = 1, \ldots, q$. From the definitions of $\hat q$ and $\hat\theta_1, \ldots, \hat\theta_q$, we can easily see that, with probability tending to $1-\gamma$, $\hat q = q$ and $\hat\theta_i - \theta_i = O_p(s_\tau)$ for $i = 1, \ldots, \hat q$. Finally, the theorem is proved by letting $\gamma \to 0$. $\square$

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