Subset Autoregression: A New Approach

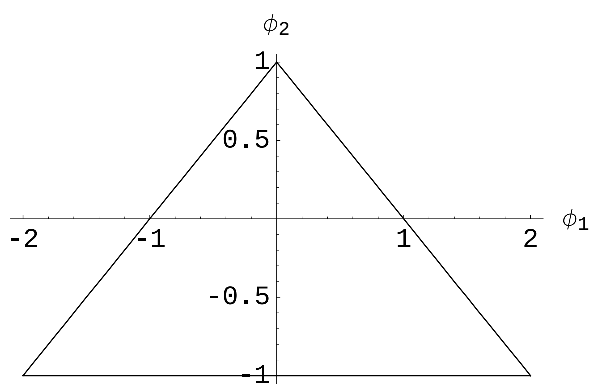
A.Ian McLeod, University of Western Ontario

Ying Zhang, Acadia University

A new family of subset autoregressive models are introduced and a comprehensive approach to model identification, estimation and diagnostic checking is developed for these models. Also a A new version of the partial autocorrelation plot is introduced. These new models are better suited to efficient model building of high-order autoregressions with long time series. Several illustrative examples are given. An R package implementation is available. In many cases subset AR models provide a useful alternative to ARMA models.

AR(p) Model Admissible Region

$x_t = \phi_1 x_{t-1} + \cdots + \phi_p x_{t-p} + a_t$, $a_t \sim \mathrm{NID}(0, \sigma_a^2)$ or $\phi(B) x_t = a_t$, B is the backshift operator on t and $\phi(B) = 1 - \phi_1 B - \ldots - \phi_p B^p$, $\Phi_p = \{(\phi_1, \ldots, \phi_p) \in \mathbb{R}^p \mid \phi(z) \neq 0,\ z \in \mathbb{C},\ |z| \leq 1\}$. AR(2) region:



■ **AR(3) Admissible Region**

Best Linear Predictor

Given an autocovariance function γ_k, $k = 0, 1, 2, \ldots$ it may easily be shown using calculus that the linear predictor $\phi_{k,1} Z_{t-1} + \ldots + \phi_{k,k} Z_{t-k}$ which minimizes the variance of the error in predicting Z_t is given by the solution to the Yule-Walker equations,

$$\Gamma_p \begin{pmatrix} \phi_{k,1} \\ . \\ . \\ . \\ \phi_{k,k} \end{pmatrix} = \begin{pmatrix} \gamma_1 \\ . \\ . \\ . \\ \gamma_k \end{pmatrix} \qquad (1)$$

where $\Gamma_p = (\gamma_{i-j})_{p\times p}$ is the covariance matrix of p successive time series values. The Durbin-Levinson is a computationally efficient and stable method of solving these special linear equations for $\phi_{k,1}, \ldots, \phi_{k,k}$.

Durbin-Levinson Algorithm

Set $\phi_{1,1} = \gamma_1 / \gamma_0$ and $v_1 = (1 - \phi_{1,1}^2)\, \gamma_0$, where v_k denotes the variance of the k step linear predictor. Then for $k = 2, 3, \ldots$ we can iteratively obtain,

$$\phi_{k,k} = (\gamma_k - \phi_{k-1,1}\, \gamma_{k-1} - \ldots - \phi_{k-1,k-1}\, \gamma_1) / v_{k-1} \qquad \text{DL-1} \qquad (2)$$

$$\begin{pmatrix} \phi_{k,1} \\ . \\ . \\ . \\ \phi_{k,k-1} \end{pmatrix} = \begin{pmatrix} \phi_{k-1,1} \\ . \\ . \\ . \\ \phi_{k-1,k-1} \end{pmatrix} - \phi_{k,k} \begin{pmatrix} \phi_{k-1,k-1} \\ . \\ . \\ . \\ \phi_{k-1,1} \end{pmatrix} \qquad \text{DL-2} \qquad (3)$$

and

$$v_k = v_{k-1}\, (1 - \phi_{k,k}^2). \qquad \text{DL-3} \qquad (4)$$

Other Applications of DL Algorithm

-parameter estimation, Yule-Walker estimates

-parameter estimation via the Burg algorithm

-test for invertibility

-reparameterization of ARMA models for exact MLE

-exact likelihood computation

-exact simulation

-Trench algorithm for efficient computation of Γ_n^{-1}

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Reparameterization

Consider an AR(p) with parameters $\phi = (\phi_1, \ldots, \phi_p)$ and let $\zeta = (\zeta_1, \ldots, \zeta_p)$, $\zeta_k = \phi_{k,k}$ where $\phi_{k,k}$, $k = 1, \ldots, p$ are the partial autocorrelations. Barndorff-Nielsen and Schou (1973) showed that $\zeta \longleftrightarrow \phi$ is a bijection which is continuous and differentiable. Hence ζ can be regarded as a reparameterization of the AR in terms of ζ. Efficient algorithms to compute the bijection $\zeta \longleftrightarrow \phi$ are based on the DL recursion.

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ARToPacf $\phi \longrightarrow \zeta$

$$\phi_{k-1,i} = (\phi_{k,i} + \phi_{k,k}\,\phi_{k,k-i}) \Big/ \left(1 - \phi_{k,k}^2\right), \quad k = p-1, \ldots, 1; \; i = 1, \ldots, k-1 \qquad (5)$$

Derivation

From DL recursion we can write,

$$\phi_{k-1,i} = \phi_{k,i} + \phi_{k,k}\,\phi_{k-1,k-i} \qquad (6)$$

by symmetry,

$$\phi_{k-1,\,k-i} = \phi_{k,\,k-i} + \phi_{k,k}\,\phi_{k-1,\,i} \tag{7}$$

Subing (8) in (7) and simplifying yields (6).

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PacfToAR $\zeta \longrightarrow \phi$

$$\phi_{k,\,i} = \phi_{k-1,\,i} - \zeta_k\,\phi_{k-1,\,k-i}; \quad k = 2, \ \ldots, \ p; \ i = 1, \ \ldots, \ k-1. \tag{8}$$

This follows directly from eqn. (4).

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Subset AR Models:

Principle of Parameter Parsimony suggests considering models such as,

$x_t = \phi_1\,x_{t-1} + \phi_2\,x_{t-2} + \phi_9\,x_{t-9} + a_t$

These models may be fit using least-squares. Least squares subset regression algorithms may be used. In general the $\mathrm{AR}_\phi(i_1,\,i_2,\,\cdots,\,i_m)$ is defined by

$$x_t = \phi_{i_1}\,x_{t-i_1} + \phi_{i_2}\,x_{t-i_2} + \cdots + \phi_{i_m}\,x_{t-i_m} + a_t \tag{9}$$

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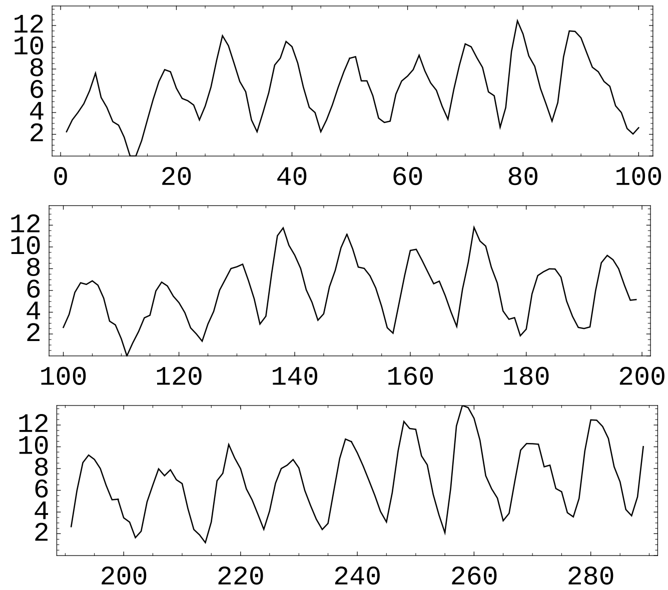
Annual Sunspot Series, 1700-1988

Consider a power transformation z^λ for $\lambda = 1,\ 0.5,\ 0.33$; $g_3 = 1.02,\ 0.18,\ -0.25$ so a square-root transformation is selected.



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AR(2) - Exact MLE

	Parameter	SE	Z
ϕ_1	1.38858	0.0425451	32.6378
ϕ_2	−0.690569	0.0425451	−16.2314
μ	48.6135	3.22231	15.0866
σ_a	16.5429		

{PortmanteauStatistic→60.4823,MaxLag→25,Pvalue→0.0000325296}

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AR$_\phi$(1, 2, 9) - LS

We fit AR$_\phi$(1, 2, i) for i = 8, 9, 10, 11, 12. Only with i = 9 was an acceptable fit obtained.

	Parameter	SE	Z
ϕ_1	1.24378	0.0588235	21.1442
ϕ_2	−0.523923	0.0938781	−5.58089
ϕ_9	0.201266	0.0588235	3.42153
μ	6.34343	0.794134	7.98786
σ_a	1.06488		

```
{PortmanteauStatistic→28.4572, MaxLag→25, Pvalue→0.16102}
```

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AR(1, 2, 4)

Is the transformation $\phi \longleftrightarrow \zeta$ useful for exact MLE of subset models?

We were able to extend the Theorem of Barndorff-Nielsen and Schou (1973) to show that the transformation $(\phi_{i_1}, \cdots, \phi_{i_m}) \longrightarrow (\zeta_{i_1}, \cdots, \zeta_{i_m})$ is a one-to-one and is continuous and differentiable. But it is very complicated and not possible to compute easily. Also the transformation is not onto.

■ Admissible Region of $AR_\phi(1, 4)$ in ζ space

■ Admissible Region of $AR_\phi(1, 2, 4)$ in ζ space

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New Subset Models

$(\zeta_{i_1}, \cdots, \zeta_{i_m}) \longrightarrow (\phi_1, \cdots, \phi_p)$, $i_m = p$, defined using `PacfToAR` given in eqn. (9) and letting $\zeta_i \in (-1, 1)$. The admissible region is simply a cube in *m*-dimensions.

And we can use the `PacfToAR` and `ARToPacf` transformations.

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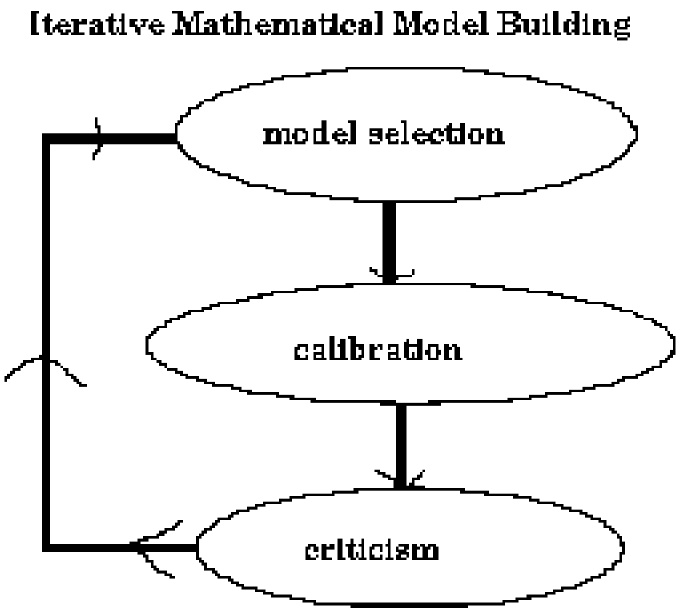
Model Building



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Exact Loglikelihood

$$L(\phi, \sigma_a^2) = -\frac{1}{2} \operatorname{Log}(\det(\Gamma_n)) - \frac{1}{2} x' \Gamma_n^{-1} x \tag{10}$$

where Γ_n is the covariance matrix of $x = (x_1, \ldots, x_n)$ and $\phi = (\phi_1, \ldots, \phi_p)$.

Champernowne (1948) showed that

$$x' \Gamma_n^{-1} x = \beta' D \beta / \sigma_a^2, \tag{11}$$

where $\beta = (-1, \phi_1, \ldots, \phi_p)$ and D is the $(p+1)$-by-$(p+1)$ matrix with (i, j)-entry

$$D_{i,j} = D_{j,i} = x_i x_j + \ldots + x_{n+1-i} x_{n+1-j}. \tag{12}$$

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Exact Loglikelihood (con't)

The standardized covariance determinant of order p, $g_p = \det(\Gamma_p / \sigma_a^2)$ where $\Gamma_p = (\gamma_{i-j})$ and $\gamma_k = \mathrm{Cov}(x_t, x_{t-k})$ may be written (Barndorff-Nielsen and Schou, 1973, eqns. 5, 8) as

$$g_p = \prod_{j=1}^{p} (1 - \zeta_j^2)^{-j}. \tag{13}$$

Hence the exact loglikelihood function (5.3.1) may now be written,

$$L(\phi, \sigma_a^2) = -\frac{n}{2} \mathrm{Log}(\sigma_a^2) - \frac{1}{2} \mathrm{Log}(g_p) - \frac{1}{2\sigma_a^2} S(\phi) \tag{14}$$

where $S(\phi) = \beta' D \beta$.

Concentrated Loglikelihood

Maximizing (5.3.1), $L(\phi, \sigma_a^2)$, over σ_a^2 , we obtain

$$\hat{\sigma}_a^2 = S(\phi) / n \tag{15}$$

and the profile loglikelihood for ϕ can be written,

$$L(\phi) = -\frac{n}{2} \mathrm{Log}(S(\phi) / n) - \frac{1}{2} \mathrm{Log}(g_p). \tag{16}$$

After the initial computation of the matrix D which only needs to be done once, each further evaluation of the likelihood $L(\phi)$ only requires $O(p^2)$ flops. Provided that $p << n$, this is much faster than other exact likelihood algorithms for ARMA

Reparameterized Concentrated Loglikelihood

To maximize $L(\phi)$, it is convenient to re-parameterize using the $\zeta = (\zeta_1, \ldots, \zeta_p)$ parameters. We can then write,

$$L(\phi(\zeta)) = -\frac{n}{2}\,\mathrm{Log}(S(\phi(\zeta))/n) - \frac{1}{2}\,\mathrm{Log}(g_p). \tag{17}$$

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The Burg Estimates

-first order efficient

-fast to compute using DL algorithm

-always in admissible region

Percival and Walden (1993, §9.5) give a new statistical derivation of the Burg algorithm.

Zhang & McLeod (2005) showed that the Burg estimates have that the first-order bias of the Burg estimates is the same as least-squares for AR(p), $p = 1, 2, 3$.

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Derivation of the Burg Estimates

In this algorithm $\vec{e}_t(k)$ and $\overleftarrow{e}_t(k)$ denote the forward and backward k-th order linear prediction errors for z_t based on $z_{t-1}, \ldots, z_{t-k}$ and $z_{t+1}, \ldots, z_{t+k}$ respectively. The Burg algorithm is characterized by the fact that $\hat{\phi}_{k,k}$ minimizes

$$\mathrm{SS}_k = \sum_{t=k+1}^{n} \vec{e}_t^{\,2}(k-1) + \overleftarrow{e}_{t-k}^{\,2}(k-1)\,. \tag{18}$$

It may be shown that the $\hat{\phi}_{k,k}$ which minimize SS_k is given by,

$$\hat{\phi}_{k,k} = A_k / B_k \tag{19}$$

$$A_k = 2\,\{\textstyle\sum_{t=k+1}^{n} \vec{e}_t(k-1)\,\overleftarrow{e}_{t-k}(k-1)\} \tag{20}$$

$$B_k = \sum_{t=k+1}^{n} \left\{\vec{e}_t^{\,2}(k-1) + \overleftarrow{e}_{t-k}^{\,2}(k-1)\right\} \tag{21}$$

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The Burg Algorithm

The Burg algorithm produces the partial autocorrelation estimates directly from the data. We assume z_t has mean zero or has been mean corrected. Then to fit an AR(p),

Step 1: Select p. Initialization. Set $k = 1$ and for $t = 2, \ldots, n$,

$$\vec{e}_t(k-1) = z_t, \tag{22}$$

$$\overleftarrow{e}_{t-1}(k-1) = z_{t-1}, \tag{23}$$

Step 2: Compute $\hat{\phi}_{k,k}$ using eqn. (11).

Step 3: Update. For $t = k+1, \ldots, n$,

$$\vec{e}_t(k) = \vec{e}_t(k-1) - \hat{\phi}_{k,k}\, \overleftarrow{e}_{t-k}(k-1), \tag{24}$$

$$\overleftarrow{e}_{t-k}(k) = \overleftarrow{e}_{t-k}(k-1) - \hat{\phi}_{k,k}\, \vec{e}_t(k-1). \tag{25}$$

Step 4: If $k = p$, terminate otherwise set $k = k+1$. Repeat Steps 2-4.

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Information Matrix for AR_ζ

Let I_ϕ and I_ζ denote the large-sample information matrix per observation for the parameters ϕ and ζ respectively in an AR(p). Then

$$I_\zeta = \mathbb{J}' \, I_{\phi(\zeta)} \, \mathbb{J}, \tag{26}$$

$$\mathbb{J} = \frac{\partial \phi}{\partial \zeta} = \prod_{k=1}^{p-1} \mathbb{J}_{p-k} \tag{27}$$

where $\mathbb{J}_{p-k}$ is derived below. In the subset case, the corresponding rows and columns of I_ζ are selected.

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Derivation of $\mathbb{J}_{p-k}$

The required Jacobian may be derived from the sequence of transformations, starting with $\phi_i = \phi_{p,i}$, $i = 1, \ldots p$ and continuing until we reach $\zeta_i = \phi_{i,i}$, $i = 1, \ldots, p$:

$$\mathbb{T}_{p-1}: \{\phi_{p,1}, \phi_{p,2}, \ldots, \phi_{p,p-1}, \phi_{p,p}\} \longleftrightarrow \{\phi_{p-1,1}, \phi_{p-1,2}, \ldots, \phi_{p-1,p-1}, \phi_{p,p}\} \tag{28}$$

$$\mathbb{T}_{p-2}: \{\phi_{p-1,1}, \phi_{p-1,2}, \ldots, \phi_{p-1,p-1}, \phi_{p,p}\} \longleftrightarrow \{\phi_{p-2,1}, \phi_{p-2,2}, \ldots, \phi_{p-2,p-3}, \phi_{p-1,p-1}, \phi_{p,p}\} \tag{29}$$

....

$$\mathbb{T}_1: \{\phi_{2,1}, \phi_{2,2}, \ldots, \phi_{p-1,p-1}, \phi_{p,p}\} \longleftrightarrow \{\phi_{1,1}, \phi_{2,2}, \ldots, \phi_{p-1,p-1}, \phi_{p,p}\} \tag{30}$$

The general form of these transformations is given by the Durbin-Levinson recursion,

$$\phi_{p,j} = \phi_{p-1,j} - \phi_{p-1,p-j}\,\phi_{p,p}, \quad j = 1, \ldots, p-1$$

Derivation of $\mathbb{J}_{p-k}$ (con't)

In general, the Jacobian of the transformation, $\mathbb{T}_{p-k}$, may be written as a partitioned matrix,

$$\mathbb{J}_{p-k} = \begin{pmatrix} J_{p-k} & A_{p-k,k} \\ 0_{p-k} & I_k \end{pmatrix} \tag{31}$$

$0_{p-k} = (0)_{k,\, p-k}$, I_k is the $k \times k$ identity matrix and $A_{p-k,k}$ is a $(p-k) \times k$ matrix whose first column is

$$\{-\phi_{p-k,p-k}, -\phi_{p-k,p-k-1}, \ldots, -\phi_{p-k,1}\}$$

and whose remaining elements are all 0. The matrix J_{p-k} may be written explicitly as the $(p-k) \times (p-k)$ matrix with (i, j)-entry $a(p, k)$ where,

$$a(p, k) = \begin{cases} 1 & \text{if } i = j \\ -\zeta_{p-k+1} & \text{if } i = p-k+1-j \wedge i \neq j \\ 1-\zeta_{p-k+1} & \text{if } i = p-k+1-j \wedge i = j \\ 0 & \text{otherwise.} \end{cases} \tag{32}$$

Example

When $p = 4$,

$$\mathbb{J}_3 = \begin{pmatrix} 1 & 0 & -\zeta_4 & -\zeta_3 \\ 0 & 1-\zeta_4 & 0 & -\zeta_2 - \zeta_1 (1+\zeta_2) \zeta_3 \\ -\zeta_4 & 0 & 1 & -\zeta_1 (1+\zeta_2) - \zeta_2 \zeta_3 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathbb{J}_2 = \begin{pmatrix} 1 & -\zeta_3 & -\zeta_2 & 0 \\ -\zeta_3 & 1 & -\zeta_1 (1+\zeta_2) & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathbb{J}_1 = \begin{pmatrix} 1-\zeta_2 & -\zeta_1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Simulation Experiment

$\zeta = \{0.5,\ 0.5,\ 0.5,\ 0.5\}$ and simulated 1,000 realizations of a time series with length $n = 1000$.

The observed sample covariance matrix of $\hat{\zeta}$ in the simulations was,

$$\begin{pmatrix} 0.0133474 & 0.00452826 & 0.00180141 & 0.000421101 \\ 0.00452826 & 0.00250702 & 0.000142564 & 0.000209351 \\ 0.00180141 & 0.000142564 & 0.000794065 & 0.0000592178 \\ 0.000421101 & 0.000209351 & 0.0000592178 & 0.000758274 \end{pmatrix}$$

and the theoretical large-sample approximation given by $I_{\zeta}^{-1} / 1000$

$$\begin{pmatrix} 0.01425 & 0.0045 & 0.0015 & 0 \\ 0.0045 & 0.00225 & 0. & 0 \\ 0.0015 & 0. & 0.00075 & 0 \\ 0 & 0 & 0 & 0.00075 \end{pmatrix}.$$

Large-Sample Distribution of MLE

Theorem 1

$\hat{\zeta} \xrightarrow{p} \zeta$ and $\sqrt{n}\,(\hat{\zeta} - \zeta) \xrightarrow{L} N(0, I_{\zeta}^{-1})$

■ **Proof**

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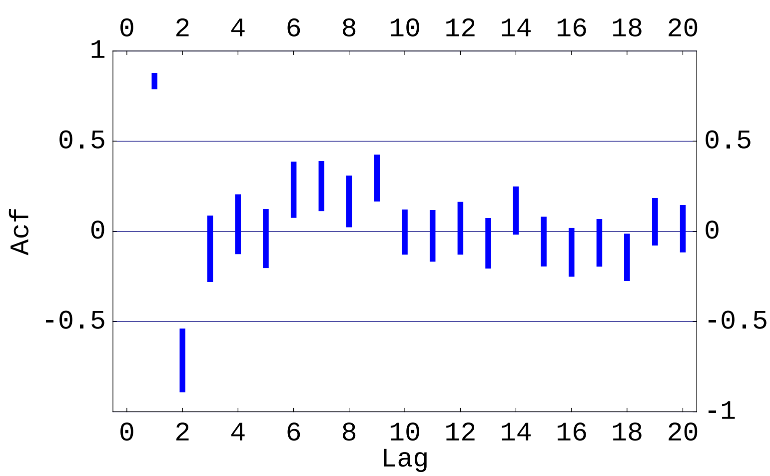
Partial Autocorrelation Plot

Using the Burg or exact MLE we obtain $\hat{\zeta}_k = \hat{\phi}_{k,k}$ and then using Theorem 1 obtain EstSd($\hat{\zeta}_k$). Plot the 95% intervals $\hat{\zeta}_k \pm 1.96\,\text{EstSd}(\hat{\zeta}_k)$.



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AIC/BIC Model Selection

For $\text{AR}_\zeta(i_1, \cdots, i_m)$, $\mathcal{L} = (-n/2)\log(\hat{\sigma}_a^2)$

Since $\hat{\sigma}_a^2 \approx c_0 \left(1 - \hat{\zeta}_{i_1}^2\right) \cdots \left(1 - \hat{\zeta}_{i_m}^2\right)$

$$\text{BIC}(i_1, \cdots, i_m) = n \log \prod_{k \in \{i_1, \cdots, i_m\}} \left(1 - \hat{\zeta}_k^2\right) + m \log(n) \tag{33}$$

We don't need to search all subsets. Just arrange $\hat{\zeta}_k^2$ in ascending order and proceed with the evaluation of the BIC.

Similarly for other IC.

Annual Sunspot Example

k	AIC	BIC	Model
1	−330.899	−327.232	{1}
2	−532.767	−525.435	{1, 2}
3	−548.387	−537.388	{1, 2, 8}
4	−561.591	−546.926	{1, 2, 8, 7}
5	−573.957	−555.625	{1, 2, 8, 7, 9}
6	−582.986	−560.987	{1, 2, 8, 7, 9, 6}
7	−588.369	**−562.704**	{1, 2, 8, 7, 9, 6, 17}
8	−590.1	−560.769	{1, 2, 8, 7, 9, 6, 17, 3}
9	**−590.659**	−557.661	{1, 2, 8, 7, 9, 6, 17, 3, 15}
10	−590.281	−553.617	{1, 2, 8, 7, 9, 6, 17, 3, 15, 18}
11	−589.845	−549.515	{1, 2, 8, 7, 9, 6, 17, 3, 15, 18, 14}
12	−589.358	−545.361	{1, 2, 8, 7, 9, 6, 17, 3, 15, 18, 14, 16}

MinAICModel → {1, 2, 8, 7, 9, 6, 17, 3, 15}

MinBICModel → {1, 2, 8, 7, 9, 6, 17}

Fitted AR_ζ for Annual Sunspots

	Parameter	SE	Z
ζ_1	0.839169	0.281783	2.97807
ζ_2	-0.671837	0.0824506	-8.14835
ζ_6	0.252221	0.0620332	4.0659
ζ_7	0.231398	0.0608168	3.80484
ζ_8	0.196152	0.0629042	3.11826
ζ_9	0.300116	0.0703228	4.26769
ζ_{17}	-0.0730557	0.0604646	-1.20824
μ	6.34343	0.657045	9.65448
σ_a	1.04209		

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Distribution Residual Autocorrelations

After fitting the residuals $\hat{a}_t$ are used to check the important assumption of independence. $\hat{r} = (r_{\hat{a}}(1), \cdots, r_{\hat{a}}(L))$

Theorem 2

$$\sqrt{n}\,\hat{r} \xrightarrow{L} N(0, \mathcal{V}), \mathcal{V} = I_m - \mathcal{X} J_\zeta I_\zeta^{-1} J_\zeta' \mathcal{X}'$$

where $\mathcal{X}$ is the $L \times m$ matrix with (i, j)-entry ψ_{i-j}, where ψ_k is the coefficient of B^k in the expansion $1/\phi(B) = 1 + \psi_1 B + \psi_2 B^2 + \ldots$ and $\phi(B) = 1 - \phi_1 B - \ldots - \phi_p B^p$.

Remark 1: Since $J_\zeta' \mathcal{X}' \mathcal{X} J_\zeta \simeq J_\zeta' I_\zeta^{-1} J_\zeta$, $\mathcal{V}$ is approximately idemptotent with rank $L - m$.

Remark 2: In the case of squared residuals, the autocorrelations are NID$(0, 1/n)$.

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Ljung-Box Test

Since $\mathcal{V}$ is approximately idemptotent with rank $L - m$ and hence we can use the Ljung-Box test

$$Q_L = n(n+2) \sum_{k=1}^{L} \hat{r}(k)^2 \Big/ (n-k) \qquad (34)$$

Under the null hypothesis of model adquacy, Q_L is approximately χ^2-distributed with $L - m$ df.

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Sunspot Series Example

Ljung-Box Test

Q_L	L	p-value
21.	20	0.05
22.3	25	0.18
24.4	30	0.33
25.5	35	0.55
30.3	40	0.55
32.	45	0.7
33.3	50	0.83

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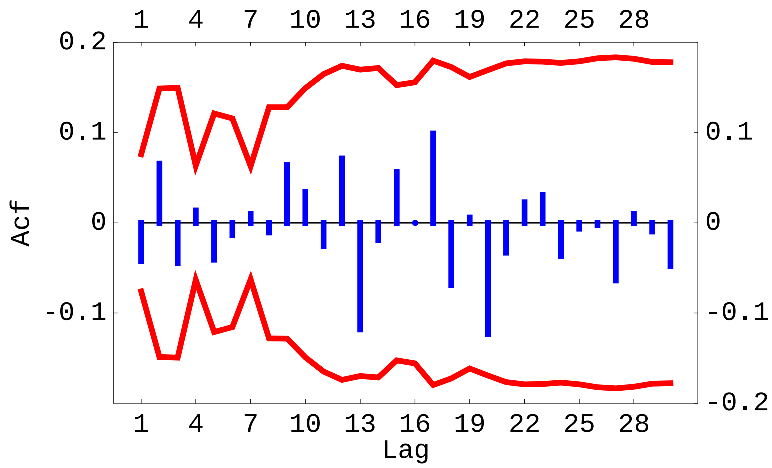
Residual Autocorrelation Plot

As noted by Hosking and Ravishanker (1993) the Bonferonni Inequality may be used to obtain 5% simultaneous significance levels.

■ **AR$_\zeta$(1, 2, 6, 7, 8, 9, 17)**



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Annual Sunspot Series

■ **Squared Residuals Ljung-Box Test**

Q_L	L	p-value
35.2	20.	0.00043
40.5	25.	0.00111
47.2	30.	0.00136
53.8	35.	0.00161
63.4	40.	0.00078
69.5	45.	0.00097
70.8	50.	0.00358

Conditional heteroscedastic variation is present. Nonlinear model needed.

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Monthly Sunspot Series, 1749-1983, $n = 2820$

Taking $L = 300$ and $M = 100$ the best AIC$_\zeta$ and BIC$_\zeta$ models were determined. The AIC and BIC were also used with the usual AR(p). The fits are summarized:

Model	IC	m	$\mathcal{L}$	AIC	BIC
AR_ζ	AIC	70	−148.2	436.4	852.6
AR_ζ	BIC	20	−236.5	513.0	631.9
AR	AIC	27	−241.1	536.3	696.8
AR	BIC	21	−252.5	547.0	671.8

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Appendix: Bonferonni Inequality

First consider white noise case, $r_k \sim \mathrm{NID}(0, 1/n)$. Then

$\Pr\left\{\left|r_k\right| < c \big/ \sqrt{n},\ k = 1, \cdots, L\right\} = 1 - \alpha$

$\Pr\{|Z_k| < c,\ k = 1, \cdots, L\} = 1 - \alpha$

where $Z_k = r_k \big/ \sqrt{n} \sim \mathrm{NID}(0, 1)$. From Basic Result in elementary probability,

$\Pr\{|Z_k| < c,\ k = 1, \cdots, L\} = 1 - \prod_{k=1}^{L} \Pr\{|Z_k| > c\}$

$(1 - 2\,(1 - \Phi(c)))^L = 1 - \alpha$

$\therefore c = \Phi^{-1}((1 + (1 - \alpha)^{1/L})/2)$

L	1	2	10	20	40	60
c	1.96	2.24	2.80	3.02	3.22	3.33

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Bonferonni Inequality

$1 \geq \Pr\{\xi_1 \bigcup \xi_2\} = \Pr\{\xi_1\} + \Pr\{\xi_2\} - \Pr\{\xi_1 \bigcap \xi_2\}$

$1 + \Pr\{\xi_1 \bigcap \xi_2\} \geq \Pr\{\xi_1\} + \Pr\{\xi_2\} = (1 - \Pr\{\overline{\xi}_1\}) + (1 - \Pr\{\overline{\xi}_2\})$

$\Pr\{\xi_1 \bigcap \xi_2\} \geq 1 - (\Pr\{\overline{\xi}_1\} + \Pr\{\overline{\xi}_2\})$

-general case established by induction

-higher-order expansion

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Bonferonni Approximation

$$\bigcap_{i=1}^{L} \Pr\{\xi_i\} \approx 1 - \sum_{i=1}^{L} \Pr\{\overline{\xi}_i\} \qquad (35)$$

taking $\xi_i = \{ |\hat{r}_i| < c \, \mathrm{EstSd}(\hat{r}_i)\}$

$\bigcap_{i=1}^{L} \Pr\{\overline{\xi}_i\} = 1 - \alpha$

$\Pr\{\overline{\xi}_i\} = \alpha / m$ so $c = \Phi^{-1}(1 - \alpha / (2 L))$

L	1	2	10	20	40	60
c	1.96	2.24	2.80	3.02	3.22	3.34

Almost but not exactly the same!