Multivariate Portmanteau Statistic

These notes provide an overview of the derivation of the portmanteau test, its theoretical optimality property and other possible alternative tests. Simulation experiments are suggested to determine empirically what works best.

For a review of multivariate vector notation see Neudecker (1969) and MathWorld articles. Also Hosking (1981B).

Monte-Carlo implementations of these tests seem to be the way to go! Asymptotic tests may in some circumstances have incorrect Type 1 errors whereas with MC tests this is more controlled. While some asymptotic tests use only the χ^2 distribution, some asymptotic tests require a much more complicated distribution which may be very tedious to program. With the advent of multi-core PC's and the parallel computing capabilities with R and *Mathematica*, computational overhead is not usually an important issue.

On the other hand, some caution is necessary because not all test statistics will be equally good. MC tests free us up from worrying about asymptotic distributions but they make the question of what is a good (powerful) test more crucial. Simultation experiments may help to clarify this situation.

- **Summary of Notation**
  - n series length
  - k multivariate dimension
  - L number of lags in portmanteau test
  - C_ℓ, $l = -L, -L+1, \ldots, 0, 1, \ldots, L$ the $k \times k$ sample autocovariance matrix
  - R_ℓ, the $k \times k$ sample autocorrelation matrix. There are 3 nonequivalent forms.
  - Q_L Box-Pierce portmanteau test. In the multivariate case, there are 4 different equivalent forms. The first three of these correspond to the 3 different (nonequivalent) defintions of R_ℓ. Essentially the normalization used makes all 3 of these equivalent as shown by Hosking. The fourth one is given by Reinsel (1997, eqn. 5.44) and was obtained by using the Chitturi (1974) definition of R_ℓ. In §1.4 below, it is shown that eqn. 5.44 of Reinsel (1997) may be obtained from the Li-McLeod definition of R_ℓ.
  - $\tilde{Q}_L$ modified portmanteau test (by context either the Li-McLeod or Hosking version)
  - D_L generalized variance whiteness test statistic. In the previous work we used $\hat{D}_L$ for the test using residuals. But I think we should drop this distinction in order to simply notation. This distinction is important when the asymptotic distributions are being derived but I don\`t think it is needed for the MC test. The multivariate extension of D_L is given below in §2.2.2. $\ddot{D}_L$ is a naive extension of D_L defined n §2.2.2. I don't recommend we bother with $\ddot{D}_L$.
  - D_L^* the generalized variance whiteness test statistic derived in the Ph.D. thesis of Lin (2006).

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1. Multivariate Portmanteau Statistic, Q_L

Various defintions for the multivariate portmanteau statistic have been suggested (**Chitturi, 1974, 1976; Hosking, 1981A; Li & McLeod, 1981**). Interestingly **Hosking (1981B)** showed that all these definitions are completely equivalent. With all these definitions the portmanteau statistic can be written,

$$Q_L = n \sum_{\ell=1}^{L} (\mathrm{vec}(R_\ell))' \left(R_0^{-1} \otimes R_0^{-1}\right) \mathrm{vec}(R_\ell) \tag{1}$$

where n is the series length, L is the maximum lag and R_ℓ is the sample autocorrelation matrix at lag ℓ. The apparent notational difference in the definitions comes from how R_ℓ is defined. Li and McLeod (1981) use the usual definition for the sample autocorelation matrix which we now define.

Let A_t, $t = 1, \ldots, n$ be k-variate white noise, that is, $A_t \sim \mathrm{NID}(0, \Delta)$, where $\Delta = \left(\sigma_{i,j}\right)_{k\times k}$. Denote the observed values of A_t by a_t, $a_t = (a_{1,t}, \ldots, a_{k,t})$, $t = 1, \ldots, n$.

In the VAR diagnostic situtation, the a_t could be the innovations for a VAR process or residuals from a fitted VAR model in which case we would normally use the notation $\hat{a}_t$. Then the sample autocovariance matrix may be written,

$$C_\ell = \left(c_{i,j}(\ell)\right) = n^{-1} \sum_{t=\ell+1}^{n} a_t\, a'_{t-\ell} \tag{2}$$

where $\ell = 0, 1, 2, \ldots$. We can write for $\ell = 0$, $C_0 = \left(c_{i,j}(0)\right) = \hat{\Delta} = (\hat{\sigma}_{i,j})_{k\times k}$. For $\ell < 0$, we define, $C_\ell = C'_{-\ell}$.

Li and McLeod (1981), take

$$R_\ell = \left(c_{i,j}(\ell) \big/ c_{i,j}(0)\right)_{k\times k} \tag{3}$$

while Chitturi (1974, 1976) takes

$$R_\ell = C_\ell\, C_0^{-1} \tag{4}$$

and Hosking (1980) takes

$$R_\ell = L'\, C_\ell\, L, \tag{5}$$

where $L\, L' = C_0^{-1}$.

It is surprising that numerically all these different definitions of $\hat{R}_\ell$ yield the same value for Q_L (Hosking 1981B). The definitions of R_ℓ are not equivalent!! Esam please study Hosking (1981B).

In the univariate case, $k = 1$, Q_L reduces to the usual portmanteau statistic,

$$Q_L = n \sum_{\ell=1}^{L} R_\ell^2 \tag{6}$$

■ 1.1 Bias Adjustment

Particularly in the case of residual diagnostic checking for fitted ARMA and VARMA models, the usual portmanteau statistic is biased. In this case, the residuals, $\hat{a}$, $t = 1, \ldots, n$ are used to compute $\hat{C}_\ell$ and $\hat{R}_\ell$ respectively. The original univariate portmanteau statistic was suggested by Box and Pierce (1970) and may be written,

$$Q_L = n \sum_{\ell=1}^{L} \hat{R}_\ell^2 \tag{7}$$

Unlike the white noise case, Box and Pierce (1970) showed that under H_0 : fitted ARMA(p, q) model is correct, $Q_L \sim \chi^2$-distributed with $L - p - q$ df. Ljung and Box (1978) suggested the modified portmanteau statistic,

$$\tilde{Q}_L = n(n+2) \sum_{\ell=1}^{L} \hat{R}_\ell^2 / (n - \ell) \tag{8}$$

Consider just the case of white noise testing,

$$\tilde{Q}_L = n(n+2) \sum_{\ell=1}^{L} R_\ell / (n - \ell) \tag{9}$$

and it is easily seen that $\in \{\tilde{Q}_L\} = L$

In fact, Davies, Triggs, and Newbold (1977, eqn. 1.7) suggested $\tilde{Q}_L$ previously and used exactly the same argument, noting that from previous results of Moran (1947, 1948),

$$\in \{R_\ell\} = \frac{n-\ell}{n(n+2)} \tag{10}$$

This formula is exact in the white noise case and an approximation in the case of residuals. It is true that Box and Ljung (1978) a fuller analysis and discussion of the modification but it is slightly surprising that statistic $\tilde{Q}_L$ is universally referred to as the Ljung-Box modified portmanteau statistic. This seems to another case of Stigler's law of eponymy (Wekipedia: http://en.wikipedia.org-/wiki/Stigler%27 s_law _of _eponymy)

In the multivariate case, the portmanteau dignostic test may be written,

$$Q_L = n \sum_{\ell=1}^{L} (\text{vec}(\hat{R}_\ell))' \left(\hat{R}_0^{-1} \otimes \hat{R}_0^{-1}\right) \text{vec}(\hat{R}_\ell) \tag{11}$$

Two modified portmanteau test statistics have been suggested. Hosking (1978) suggested,

$$\tilde{Q}_L = n(n+2) \sum_{\ell=1}^{L} (\text{vec}(\hat{R}_\ell))' \left(\hat{R}_0^{-1} \otimes \hat{R}_0^{-1}\right) \text{vec}(\hat{R}_\ell) \big/ (n-\ell) \tag{12}$$

while Li and McLeod (1981) suggested,

$$\tilde{Q}_L = n \sum_{\ell=1}^{L} (\text{vec}(\hat{R}_\ell))' \left(\hat{R}_0^{-1} \otimes \hat{R}_0^{-1}\right) \text{vec}(\hat{R}_\ell) + k^2 \frac{L(L+1)}{2\,n} \tag{13}$$

Both Li and McLeod (1981) and Hosking (1981) provided simulation experiments to demonstrate the improvement of their suggested modified portmanteau test. Li (2004, p.25) notes that Ledholter (1983) compared these two modified tests with the original Q_L and found that both modifications worked equally well and were better than Q_L. Kheoh & McLeod (1992) suggested that the variance of the Li-McLeod modified portmanteau test was less and provided simulation evidence that in the univariate case there was a slight improvement in power.

In the univariate case McLeod (1978) showed that $\hat{r} \sim N(0, \text{Ᵽ})$, where Ᵽ is given in the paper. Using this fact Ljung (1986) examined a test based on the exact value of the asymptotic distribution obtained by numerical inversion of the characteristic function. Minor improvements were noted.

■ 1.2 Notes on the Derivation of Multivariate Portmanteau Statistic

It can be seen that the distributions of C_ℓ and R_ℓ both depend on Δ. Under the null hypothesis of white noise, it may be shown that vec R_ℓ is approximately NID with mean 0 covariance matrix $\Delta \otimes \Delta$. From this it follows that $Q_L \sim \chi^2_\nu$, where $\nu = k^2 L$.

Without loss of generality, we can work with C_ℓ and assume that innovation covariance matrix is in correlation form and treat the case of known $\sigma_{i,i} = 1$, that is,

$$R_\ell = C_\ell = \left(c_{i,j}(\ell)\right) = n^{-1} \sum_{t=\ell+1}^{n} a_t\, a'_{t-\ell}.$$

$$\text{Var}\left(c_{i,j}(\ell)\right) = n^{-2} \sum \text{Var}\left(a_{i,t}\, a_{j,t-\ell}\right) = n^{-2} \sum \in \left\{\left(a_{i,t}\, a_{j,t-\ell}\right)^2\right\}$$

since $\mathbb{E}\left\{a_{i,t}\, a_{j,t-\ell}\right\} = 0$ for $\ell \neq 0$ and using the following results.

Recall the fourth moment result, provided all moments exist, (Hannan, 1970, p.23; Brillinger, 1981, §2.3; Isserlis, 1918)

$\mathbb{E}\{X_1 X_2 X_3 X_4\} = \mathbb{E}\{X_1 X_2\}\, \mathbb{E}\{X_3 X_4\} + \mathbb{E}\{X_1 X_3\}\, \mathbb{E}\{X_2 X_4\} + \mathbb{E}\{X_1 X_4\}\, \mathbb{E}\{X_2 X_4\} + \mathrm{cum}(X_1, X_2, X_3, X_4)$

where $\mathrm{cum}(X_1, X_2, X_3, X_4)$ denotes the cumulant which is zero if (X_1, X_2, X_3, X_4) is multivariate normal.

$$\begin{aligned}
&\mathbb{E}\left\{\left(a_{i,t}\, a_{j,t-\ell}\right)^2\right\} \\
&= \mathbb{E}\left\{a_{i,t}\, a_{j,t-\ell}\, a_{i,t}\, a_{j,t-\ell}\right\} \\
&= \mathbb{E}\left\{a_{i,t}\, a_{j,t-\ell}\right\} \mathbb{E}\left\{a_{i,t}\, a_{j,t-\ell}\right\} + \mathbb{E}\{a_{i,t}\, a_{i,t}\}\, \mathbb{E}\left\{a_{j,t-\ell}\, a_{j,t-\ell}\right\} + \mathbb{E}\left\{a_{i,t}\, a_{j,t}\right\} \mathbb{E}\left\{a_{i,t}\, a_{j,t}\right\} \\
&= \sigma_{i,i}\, \sigma_{j,j}
\end{aligned}$$

Hence we have

$n\, \mathrm{Var}\left(c_{i,j}(\ell)\right) = \sigma_{i,i}\, \sigma_{j,j}$

Similarly we can show that,

$n\, \mathrm{Cov}\left(c_{g,h}(\ell),\, c_{i,j}(\ell)\right) = \sigma_{g,i}\, \sigma_{h,j}$

$$\begin{aligned}
&\mathrm{Cov}\left(c_{g,h}(\ell),\, c_{i,j}(\ell)\right) = n^{-2} \sum \mathrm{Cov}\left(a_{g,t}\, a_{h,t-\ell},\, a_{i,t}\, a_{j,t-\ell}\right) \\
&= n^{-2} \sum \mathbb{E}\left\{a_{g,t}\, a_{h,t-\ell}\, a_{i,t}\, a_{j,t-\ell}\right\}
\end{aligned}$$

Hence we can write, $\mathrm{Cov}(\mathrm{vec}(R_\ell)) = \Delta \otimes \Delta$

and more generally,

$\mathrm{Cov}((\mathrm{vec}(R_1), \ldots, \mathrm{vec}(R_L))) = 1_L \otimes \Delta \otimes \Delta$

In the case of testing for whiteness, the test statistic,

$$Q_L = n \sum_{\ell=1}^{L} (\mathrm{vec}(R_\ell))' \left(\Delta^{-1} \otimes \Delta_0^{-1}\right) \mathrm{vec}(R_\ell) \tag{14}$$

is asymptotically χ^2 with df $k^2 L$. In practice Δ is not known but the result still holds if the usual moments estimate of Δ is used.

■ 1.3 Optimal Properties of Q_L

It is well-known that the likelihood ratio test enjoys many optimal properties. Whittle (1952) suggests testing the goodness of fit of an AR(p) by using a likelihood-ratio test for an AR($p + L$) for sufficiently large L. More generally if L is large enough, the likelihood-ratio statistic based on an ARMA(p, q) vs. AR(L) is still approximately χ^2 on df $= L - p - q$ (Whittle, 1952).

Hosking (1978) shows that in the univariate case, the Q_L is related to this test. Specifically Q_L is equivalent to a Lagrange multiplier test of

$\mathcal{H}_0$: AR(p) vs. $\mathcal{H}_1$: constrained AR($p + L$)

More precisely, the model under $\mathcal{H}_1$ is an AR(L) process which is fitted to the residuals in the AR(p). Thus the alternative process can be written (Hosking, 1978, eqn. 6),

$$\phi(B)\, \alpha(B)\, z_t = a_t \tag{15}$$

where $\phi(B)$ and $\alpha(B)$ are polynomials of degrees p and L respectively.

To the extent that $\mathcal{H}_1$ provides a reasonable model, we must agree that Q_L seems like a good idea for a test statistic.

In the univariate case, Newbold (1980) shows that the Lagrange multiplier test for testing ARMA(p, q) vs ARMA($p+L$, q) is equivalent to a test based on the first L residual autocorrelations,

$\hat{r} = \left(\hat{R}_1, \ldots, \hat{R}_L\right)'$.

The test statistic derived by Newbold (1980) can be written,

$S = \hat{r}' \mathcal{P}^{-1} \hat{r}$,

where $\mathcal{P}$ is the covariance matrix given by McLeod (1978). For finite L, $\mathcal{P}$ is nonsingular but for large L it is approximately indempotent and hence the usual portmanteau test may be derived using this fact.

Hosking (1981A) derives the multivariate portmanteau test using Q_L using a Lagrange multiplier test approach. Since it is generally well known that the Lagrange multiplier test is asymptotically equivalent to the likelihood-ratio test, this provides further support to using Q_L.

Cox and Hinkley (1974) provide an overview of likelihood-ratio tests and Lagrange multipier test and show that these two tests are asymptotically equivalent.

In summary, the portmanteau test based on Q_L has strong theoretical justification in that it approximates in a general way a likelihood-ratio test. Nevertheless, it has been established that tests based on the generalized variance of the standardized residuals typically perform with higher power.

■ 1.4 Computation of Q_L

The usual definition of the multivariate portmanteau statistic,

$$Q_L = n \sum_{h=1}^{L} \operatorname{tr}\left(C_h' C_0^{-1} C_h C_0^{-1}\right) \tag{16}$$

Computationally, we can obtain $U = C_h' C_0^{-1}$ and $V = C_h C_0^{-1}$ and then

$$Q_L = n \sum_{i,j} U_{i,j} V_{j,i} \tag{17}$$

where we have used a well-known result for the trace of a matrix product (*MathWorld*).

Reinsel (1993, eqn. 5.36) and Reinsel (1997, eqn. 5.44) defines $Q(L)$ in an equivalent but notationally different way. His eqn. (5.44) in Reinsel (1997) is in our notation,

$$Q_L = n \sum_{\ell=1}^{L} \sum_{i=1}^{k} \sum_{j=1}^{k} r_{i,j}(\ell)\, r_{i,j}(-\ell) \tag{18}$$

where, taking $\ell > 0$ in both of the following expressions, $\left(r_{i,j}(\ell)\right)_{k\times k} = C_\ell C_0^{-1} = R_\ell$ and for $\left(r_{i,j}(-\ell)\right)_{k\times k} = C_\ell' C_0^{-1} = R_{-\ell}$, where the Chitturi (1974) definition of multivariate residual autocorrelations is used. The above equation for Q_L provides a fourth equivalent defintion.

2. Other Multivariate Portmanteau Tests

2.1 A Simplified Alternative to Q_L

Lin (2006, eqn. 2.21) suggested a simplifed version of the portmanteau test. In the whiteness testing problem, this test can be written,

$$\begin{aligned} Q_L^{*} &= n \sum_{\ell=1}^{L} \left(\text{vec}(\hat{R}_\ell)\right)' \text{vec}(\hat{R}_\ell) \\ &= n \sum_{\ell=1}^{L} \sum_{i=1}^{k} \sum_{j=1}^{k} r_{i,j}(\ell)\, r_{i,j}(-\ell) \\ &= \sum_{m=1}^{k} Q_L^{(m)} + n \sum_{i \neq j} r_{i,j}(\ell)\, r_{i,j}(-\ell) \end{aligned} \tag{19}$$

where $Q_L^{(m)}$ is the usual univariate portmanteau test statistic for the m-th series, $m = 1, \ldots, k$. The second term in the above equation was suggested by McLeod (1979) as a test for causality between series i and series j in the presence of instantaneous causality and its distribution was obtained. In the more general case in eqn. (4), the asymptotic distribution of the cross-product term has been derived by Lin (2006, §2.3, Lemma2, page 26).

The asymptotic distribution of Q_L^{*} has not been derived but using the methods given in Lin (2006, §2.3) this would not be difficult. But this distribution is quite complex and so not likely to useful unless suitable approximations were found.

The test statistic Q_L^{*} does not have as strong a justification as the usual portmanteau test which can be argued approximates a likelihood-ratio test.

2.2 Generalized Variance Tests

2.2.1 Univariate Case

For the univariate case, **Peña and Rodriguez (2002)** suggested that the generalized variance of the standardized residuals residuals might be more powerful in many situations. **Lin and McLeod (2006)** showed that the original formulation of the test (Peña and Rodriguez, 2002) had many problems and these problems could be overcome by resorting to a Monte-Carlo version of the test. Lin and McLeod (2006) also provided convincing evidence that the Monte-Carlo test worked better than the asymptotic version of the portmanteau test as well as Monte-Carlo test using the univariate Q_L. Lin and McLeod (2008) extended the diagnostic test to ARMA with infinite variance innovations. In the univariate white-noise case,

$$D_L = n\left(1 - \det(\text{toe}(R_1, \ldots, R_L))^{1/L}\right) \tag{20}$$

where det(•) is the determinant and toe(•) turns a vector into a Toeplitz matrix. For diagnostic checking, we use residuals,

$$\hat{D}_L = n\left(1 - \det\left(\text{toe}\left(\hat{R}_1, \ldots, \hat{R}_L\right)\right)^{1/L}\right) \tag{21}$$

Note the normalization is used so that D_L and $\hat{D}_L$ converge in distribution for large samples. For Monte-Carlo tests, the normalization is not needed.

2.2.2 Multivariate Case

The simplest extension to multivariate diagnostic checking would be,

$$D_L = n\left(1 - \det\left(\mathrm{toe}\left(\mathrm{vec}\,\hat{R}_1,\ \ldots,\ \mathrm{vec}\,\hat{R}_L\right)\right)^{1/L}\right) \tag{22}$$

Three different, non-equivalent versions, of D_L may be defined corresponding to the three different definitions of R_ℓ. Using the Chitturi definition, the distribution of D_L does not depend on Δ in the white noise case (please provide a more detailed proof of this). If the Li-McLeod version R_ℓ is used, we denote the above statistic by $\ddot{D}_L$ and the distribution of this statistic obviously depends on Δ.

This statistic depends on the nuisance parameters, Δ. I think it is likely the case that for MC testing this is not important. Another previous consideration by Lin (2006) was that it is not clear how the asymptotic distribution could be obtained.

For these reasons a new generalized variance test for the multivariate case was developed by Lin (2006) and Lin (2006) presented a heurestic argumentthat an approximation to the above statistic is given by

$$D_L^{*} = \sum_{m=1}^{k} \hat{D}_L^{(m)} \; + \; n \sum_{i \neq j} r_{i,j}(\ell)\, r_{i,j}(-\ell) \tag{23}$$

An important consideration in choosing to work with $\hat{D}_L^{*}$ is that the asymptotic distribution can be obtained (Lin, 2006). But unfortunatelythis asymptotic distribution does not seem to be useful because of large size distortions in the test and it is also difficult to compute. For this reason, D_L^{*} may not be of much interest, I think.

Some simulation studies reported in Lin (2006) and Lin and McLeod (2009) lend support to the idea that perhaps $\hat{D}_L^{*}$ will perform better than the usual Li-McLeod portmanteautest. Even for the MC test version of this statistic.

In these simulation experiments VAR(1) was fit to several types of VARMA$(p,\ q)$ models where $0 \le p \le 1$ and $0 \le q \le 1$ and $k = 1$. The innovation covariance matrix was nondiagonal with unit variances and off-diagonal element 0.71.

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3. Simulation of VAR and VARMA

Because the VARMA model is too complex and difficult to estimate and there is no software for VARMA estimation available in R, we will focus on VAR diagnostic checking. When considering power we may consider higher order VAR, seasonal VAR as well as VARMA series. In general the VARMA$(p,\ q)$ model may be written,

$$\phi(B)\, Z_t = \zeta + \theta(B)\, A_t \tag{24}$$

where $A_t \sim \mathrm{NID}(0,\ \Delta)$, where Δ is the $k \times k$ covariance matrix. Often it will may be the case that Δ is nondiagonal and so we say that instantaneous causality is present (McLeod, 1979 and references to Pierce, 1977 and Haugh and Pierce, 1977). The AR and MA components are respectively $\phi(B) = 1_k - \phi_1 B - \ldots - \phi_p B^p$ and $\theta(B) = 1_k - \theta_1 B - \ldots - \theta_q B^q$ where 1_k denotes the $k \times k$ identity matrix and $\phi_1,\ \ldots,\ \phi_p,\ \theta_1,\ \ldots,\ \theta_q$ are all $k \times k$ matricies. It is essential that eqn (24) represent a stationary time series. Sometimes homogenousnonstationaryprocesses may be allowed but explosive time series are unrealistic and may cause numeric problems so they should be avoided. A process is said to be explosive if $\det(\phi(z)) = 0$ has a root inside the unit circle and it is homogenousnonstationaryor stationary when $\det(\phi(z)) = 0$ has all roots on or outside the unit circle.

For our simulation functionwe will assume stationarity so that $\det(\phi(z)) = 0$ has all roots outside the unit circle.

The VARMA$(p,\ q)$ process in (24) may be simulated recursively once initial values have been determined. Many previous authors have simply used (24) to generate a series of length $n + \mathfrak{L}$ and then deleted the first $\mathfrak{L}$ values for warmup. A better approach is to obtain the initial values using a high order MA approximation,

$$Z_t \doteq \zeta + A_t + \psi_1 A_{t-1} + \ldots + \psi_{\mathfrak{L}} A_{t-\mathfrak{L}} \tag{25}$$

where the ψ_h, $h = 1, 2, \ldots$ are determined recursively by solving the identity,

$\phi(B) \psi(B) = \theta(B)$

So

$\psi_h = -\theta_h + \phi_1 \psi_{h-1} + \ldots + \phi_p \psi_{h-p}$

where we define $\theta_0 = -1_k$, $\theta_h = 0$ for $h < 0$ and similarly $\psi_0 = 1_k$ and $\psi_h = 0$ for $h < 0$.

This simulation is typically more accurate. If stationarity is assumed then $\psi_h \to 0$ has $h \to \infty$. In practice we choose $\mathfrak{L}$ so that $\psi_{\mathfrak{L}}$ is negligible, $\|\psi_L\| < \epsilon$, where the matrix norm is indicated and ϵ is a small positive number.

Using eqn. (25) we can obtain starting values, when $q > 0$, Z_p, Z_{p-1}, $\ldots$, Z_1, A_p, $\ldots$, A_{p-q+1} and when $q = 0$ we only need, Z_p, Z_{p-1}, $\ldots$, Z_1.

Previously in the gvtest package, I used C code to compute the recursions involved in eqns. (24) and (25) in the univariate case. For maximumefficiency this needs to be extended to the multivariate case.

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4. Further Work Needed

We will focus entirely on checking the adequacy of VAR models. Perhaps we should also look at the white-noise testing problem as well since it is relatively simple. In the univariate case, the generalized variance test worked better than the usual portmanteau test but less spectacularly so than in the residual diagnostic checking problem (Lin 2006).

Our simulations should focus on the empirical power. It would be nice if we parameterize the alternative model in the simulation so that one particular parameter setting corresponds to the original model since the power in this case is the type 1 error rate. In this way we can easily compare the size of the test as well. We have already noted that MC tests usually have the correct size but this is sometimes not the case for asymptotic tests.

I am currently developing an R package to implement multivariate portmanteau tests. I am making use of previous R code from Jen-Wen Lin but I have made some improvements on the efficiency. I like the simulation for AR that was developed. I am thinking of using C code for some of the loops to speed it up.

Monte-Carlo testing can very easily be implemented using the software Rmpi and so it can be run effectively on multi-core CPU as well as on departmental cluster. This means we can do a lot more simulation.

It seems unlikely that $\hat{D}_L^*$ or even D_L is really going to be optimal in all situations relative to Q_L.

We need to be very careful with the design of our simulation study. We need to pay attention of how we are going to present the results. We need to investigate how the correlation structure, Δ, affects the tests. Most simulations can be done with $k = 2$ but we should try some simulation with $k = 3, 4, 5$ just to check how the test is effected by dimension.

We should also be sure to include models with seasonal lags corresponding to $s = 12$ for monthlydata.

Another concern is that the models should be stationary (and perhaps identifiable also). Identifiability in the case of VARMA models is very complex -- see Hannan (1979) for example.

In the case of the k-dimensional VAR(p) model, $\phi(B) Z_t = A_t$, where $\phi(B) = 1 - \phi_1 B - \ldots - \phi_p B$, $A_t \sim \text{NID}(0, \Delta)$, where each of $\phi_1, \ldots, \phi_p$ are $k \times k$ matricies, the stationarity condition is that all zeros of the equation, $\det(\phi(z)) = 0$ are outside the unit circle, that is, $|z| > 1$. For the VAR(1) case, this means that all eigenvalues of ϕ_1 are less than 1 in absolute value. But when $p > 1$, the stationarity condition is not so easily determined computationally. This suggests using an approach such as **Newton (1982)** or perhaps **Tiao and Tsay (1989)** for simulating the alternative higher order VAR process.

5. References

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