Rank Test for Trend in Time Series

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Outline

Background

- When using the usual Mann-Kendall tau to test for trend, the data need to be i.i.d (Hipel and McLeod, 1994, Chapter 23).
- Real-world data do not always meet this requirement.
- El-Shaarawi & Niculescu (1992) derived an expression for the exact variance of the Mann-Kendall tau in the case of MA(1) and MA(2) errors and modified the test for such data.
- The Mann-Kendall tau needs to be studied further in more general error models.

Time Series Process with a Deterministic Trend

- The observations of a time series exhibit a trend of the form

$$X_t = M(t) + \mu_t,$$

where $M(t)$ is a monotonic trend and $\{\mu_t\}$ is a zero mean stationary process

- Null hypothesis for testing trend

$$H_0 : M(t) = M_0, wlog, M_0 = 0$$

U-Statistics

- Define the *U*-statistic

$$U_n = \frac{1}{\binom{n}{k}} \sum_{(n,k)} h(X_{i_1}, \ldots, X_{i_k}),$$

  where $\sum_{(n,k)}$ is taken over all subsets $1 \leq i_1 < \ldots < i_k \leq n$ of $1, 2, \ldots n$, and k is called the degree of the kernel $h(X_{i_1}, \ldots, X_{i_k})$.

- *U*-statistics are unbiased estimators of parameters in the IID data case,

$$\theta = E[h(X_{i_1}, \ldots, X_{i_k})]$$

  .

Kendall's Tau Statistic

For bivariate observations, let $A_i = (X_i, Y_i)$, $i = 1 \ldots n$ be continuous.

$$h(A_i, A_j) = \mathrm{sgn}(X_j - X_i)\mathrm{sgn}(Y_j - Y_i).$$

Define

$$\widehat{Z}_n = \frac{1}{\binom{n}{2}} \sum_{i<j}^{n} h(A_i, A_j)$$

a U-statistic with a symmetric kernel $h(A_i, A_j)$, which is an unbiased estimator of $\theta = E[h(A_i, A_j)]$ for IID data.

Mann-Kendall Tau Statistic, $\widehat{\tau}_n$

In the case of testing for (positive) trend, Y_i is replaced by i, $i = 1, \ldots, n$. The Mann-Kendall statistic is denoted by $\widehat{\tau}_n$, which is a U-statistic with a non-symmetric kernel

$$h(X_i, X_j) = \mathrm{sgn}(X_j - X_i)$$

Test statistic,

$$\widehat{\tau}_n = \frac{1}{\binom{n}{2}} \sum_{i<j}^{n} \mathrm{sgn}(X_j - X_i)$$

which is a special version of the Kendall correlation.

Error Term, μ_t

A covariance stationary process $ARMA(p, q)$ may be written as

$$\begin{aligned}\mu_t =& \rho_1 \mu_{t-1} + \rho_2 \mu_{t-2} + \ldots + \rho_p \mu_{t-p} \\ &+ \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} - \ldots - \theta_q \varepsilon_{t-q}\end{aligned}$$

where ε_t are i.i.d. noises with $E(\varepsilon_t) = 0$ and $\text{Var}(\varepsilon_t) = \sigma_\varepsilon^2$.

Error Term, μ_t (Conti.)

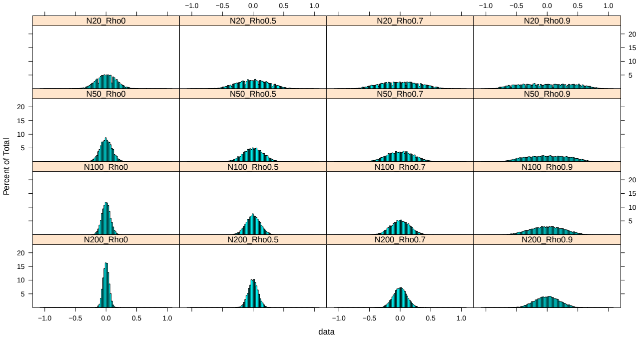
Suppose that under the null hypothesis of no trend $X_t = M_0 + \mu_t$, the error terms, $\{\mu_t\}$, are $AR(1)$ or $MA(1)$ processes.

$$AR(1) : \mu_t = \rho\mu_{t-1} + \varepsilon_t$$

$$MA(1) : \mu_t = \varepsilon_t - \rho\varepsilon_{t-1}$$

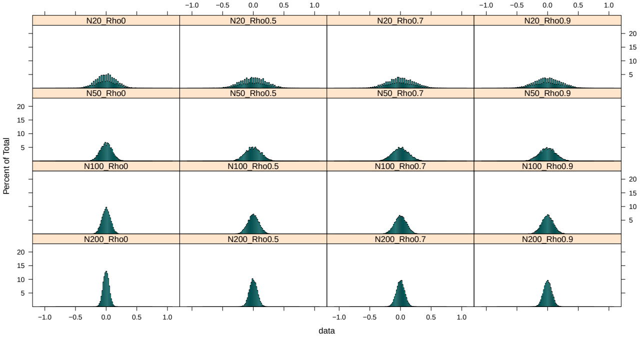
where ε_t are independent random noise terms from a standard normal distribution, $t \in \{1, \ldots, n\}$ where n is the length of the time series and $|\rho| < 1$

The simulated null distribution[1] of $\widehat{\tau}_n$ for time series with errors following a normal $AR(1)$ process

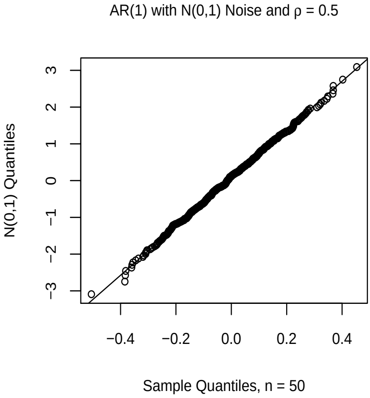


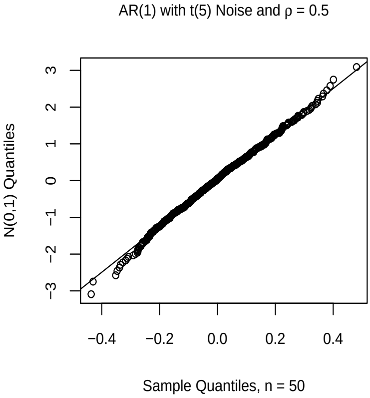
[1] The histogram plots were generated by 10,000 realizations.

The simulated null distribution of $\widehat{\tau}_n$ for time series with errors following a normal $MA(1)$ process



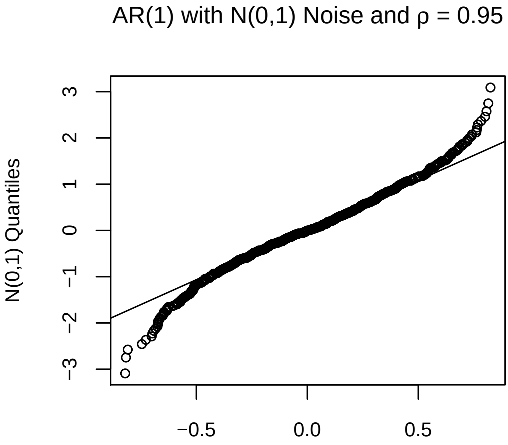
Probability Plots [2] of $\widehat{\tau}_n$





[2] The QQ plots were generated by 500 realizations.

Probability Plots of $\widehat{\tau}_n$ (conti.)



Simulation Results of the Null Distribution of $\widehat{\tau}_n$

- $\widehat{\tau}_n$ converges to the normal distribution faster in MA than in AR.
- $\widehat{\tau}_n$ converges to the normal distribution faster when the autocorrelation is weaker.
- Normality can hardly be achieved when the time series is nearly nonstationary.
- Normality is robust in terms of the noise distribution.

CLT in the Independent Case

If $\{X_i\}$ are i.i.d., under the null hypothesis of no trend, it is well known that as $n \to \infty$

$$\sqrt{n}\widehat{\tau}_n \xrightarrow{D} N(0, 4\xi_1)$$

where

$$h_1(a_1) = E[h(a_1, A_2)]$$

$$\xi_1 = \text{Var}[h_1(A_1)] = \frac{1}{9}$$

Weakly Dependent Case: Strong Mixing

Suppose $\{X_t\}$ is a stationary sequence defined on a probability space $(\Omega, \mathcal{B}, \mathcal{P})$ and $\mathcal{F}_n^m = \sigma\{X_t : n \leq t \leq m\}$ is the σ-algebra generated by $(X_n, \ldots, X_m)$. For $n \geq 1$, define

$$\beta(m) = E\{\sup_{A \in \mathcal{F}_{n+m}^{\infty}} |P(A|\mathcal{F}_0^n) - P(A)|\}$$

$\{X_t\}$ is absolutely regular (or β) mixing if the coefficient $\beta(m) \to 0$, as $m \to \infty$.

CLT in the Weakly Dependent Case

- Yoshihara (1976) proves the CLT for a *U*-statistic on a stationary absolutely regular process whose rate of convergence of $\beta(m)$ to 0 is $O(m^{-(2+\delta)/\delta})$, for some $\delta > 0$, with additional conditions on the kernel of the U-statistic.
- Mokkadem (1988) shows that stationary ARMA processes are absolutely regular, with $\beta(m) = O(r^m)$ for some $0 < r < 1$, so that the rate of convergence of $\beta(m) \to 0$ satisfies Yoshihara's condition.

CLT in the Weakly Dependent Case (Conti.)

In the no trend case, if $\{\mu_t\}$ are stationary absolutely regular, the $\widehat{\tau}_n$ satisfies the kernel conditions, so that if the rate of convergence condition of Yoshihara (1976) is met, then

$$\sqrt{n}\widehat{\tau}_n \xrightarrow{D} N(0, 4\sigma^2)$$

$$\sigma^2 = \sigma_1^2 + 2\sum_{s=1}^{\infty} \sigma_{1,s}^2$$

$$\sigma_1^2 = \mathrm{Var}[h_1(X_1)]$$

$$\sigma_{1,s}^2 = \mathrm{Cov}[h_1(X_t), h_1(X_{t+s})].$$

Lemma 1: CLT of Mann-Kendall Tau in the ARMA Error Case

In the no trend case, if $\{\mu_t\}$ is a zero mean stationary ARMA process,

$$\frac{\sqrt{n}}{2\sigma}\hat{\tau}_n \xrightarrow{\mathcal{D}} N(0,1).$$

Bootstrapping U-Statistics[3] for Weakly Dependent Data

Dehling and Wendler (2010) show that when the conditions of Yoshihara (1976) are met, the block bootstrapped U-statistic, follows a CLT, with

$$|\,\mathrm{Var}^*(\sqrt{bl}U_n^*) - \mathrm{Var}(\sqrt{n}U_n)| \xrightarrow{a.s.} 0,$$

and

$$\sup_x |P^*[\sqrt{bl}U_n^* - E^*[U_n^*]) \leq x] - P[\sqrt{n}(U_n - \theta) \leq x]| \xrightarrow{a.s.} 0.$$

where b = number of blocks, and l = length of each block.

$$bl \approx n$$

[3] The superscript * refers to bootstrap.

Lemma 2: Bootstrapping Mann-Kendall Tau Statistics in the ARMA Error Case

In the no trend case, if $\{\mu_t\}$ is a zero mean stationary ARMA process,

$$P\left[\hat{\tau}_n \leq \tau_0\right] \sim \Phi\left(\tau_0 / \sqrt{\frac{bl}{n} Var^*\left(\hat{\tau}_n^*\right)}\right) \quad (1)$$

$$P\left[\hat{\tau}_n \leq \tau_0\right] \sim P^*\left[\sqrt{\frac{bl}{n}}\left(\hat{\tau}_n^* - E^*\left[\hat{\tau}_n^*\right]\right) \leq \tau_0\right] \quad (2)$$

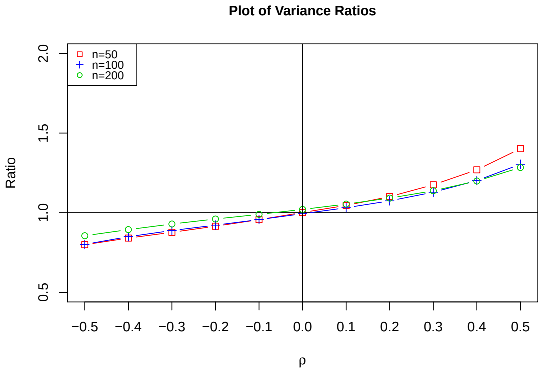
where $\hat{\tau}_n^*$ is the circular block or moving block bootstrap MK statistics; τ_0 is the observed MK statistic.

Choosing Block Length

- The optimal block length strongly depends on the correlation structure.
- We have chosen $n^{1/3}$ for time series data with weak and simple correlation structure such as AR(1) with moderate ρ values; $\sqrt{n}$, otherwise, such as AR(2).
- The effects of autocorrelation on the variance estimate were examined by comparing the exact variance of $\widehat{\tau}_n$ to a sample mean of $\hat{\tau}_n^*$ based on 1000 simulations of normal AR(1) processes, and block length $l = n^{1/3}$.

$$\text{Variance Ratio} = \frac{\text{Var}(\hat{\tau}_n)}{\text{a sample mean of Var}^*(\hat{\tau}_n^*)}$$

Effects of Autocorrelation on Variance Estimate



The bootstrap variance estimation underestimates (overestimates) the true variance when $\rho > 0$ ($\rho < 0$).

Approximating the Null Distribution of $\widehat{\tau}_n$

Eqns. (1) and (2) in Lemma 2 provides two block bootstrap methods:

- Bootstrap Distribution Approximation (*Bootprob*);
- Normal Approximation with Bootstrap Variance Estimate(*Norm(app)*).

Bootprob and *Norm(app)* were compared with the Monte Carlo test (*MC*) and the null distribution of $\hat{\tau}_n$ assuming the process is *IID*.

Type I Error Comparison[4] for $AR(1)$ Norm Noise Case, $n = 50$, block size $l = n^{1/3}$

ρ	Size	*MC*	*Bootprob*	*Norm(app)*	*IID*
0	10%	0.095	0.102	0.099	0.093
	5%	0.047	0.058	0.055	0.049
	1%	0.008	0.016	0.013	0.010
0.1	10%	0.104	0.114	0.111	0.120
	5%	0.053	0.069	0.065	0.070
	1%	0.011	0.024	0.021	0.017
0.2	10%	0.106	0.123	0.119	0.145
	5%	0.055	0.074	0.070	0.088
	1%	0.011	0.029	0.025	0.028
0.3	10%	0.104	0.127	0.122	0.168
	5%	0.053	0.079	0.073	0.110
	1%	0.009	0.030	0.025	0.040
0.4	10%	0.108	0.140	0.136	0.195
	5%	0.057	0.091	0.084	0.137
	1%	0.011	0.042	0.033	0.060
0.5	10%	0.105	0.150	0.146	0.220
	5%	0.054	0.100	0.092	0.161
	1%	0.009	0.048	0.038	0.078

[4] The simulated significance level was based on 10,000 tests.

Type I Error Comparison for $AR(2)$ Norm Noise Case, $n = 200$, block size $l = n^{1/2}$

(ϕ_1, ϕ_2)	Size	*MC*	*Bootprob*	*Norm(app)*	*IID*
$(0.5, 0.25)$	10%	0.109	0.158	0.156	0.328
	5%	0.058	0.107	0.101	0.283
	1%	0.012	0.053	0.042	0.206
$(1, -0.25)$	10%	0.108	0.133	0.132	0.295
	5%	0.054	0.084	0.079	0.241
	1%	0.009	0.030	0.024	0.159
$(1.5, -0.75)$	10%	0.105	0.098	0.097	0.179
	5%	0.051	0.049	0.049	0.124
	1%	0.008	0.011	0.010	0.050
$(1, -0.6)$	10%	0.102	0.101	0.100	0.114
	5%	0.051	0.052	0.052	0.062
	1%	0.010	0.012	0.012	0.016

Type I Error Comparison Results

- The *MC* method achieves significance levels very close to the nominal levels.
- The two block bootstrap methods show similar tendencies for Type I error inflation. However the inflation levels are consistently smaller than those for the *IID* estimates, particularly as correlation increases.
- Larger sample sizes reduce the levels for *Bootprob* and *Norm(app)*, which is not the case for the *IID* levels.
- Larger sample sizes and proportionally large block sizes may be required for more complex correlation structures.

Power Comparison

Trend decomposition is the necessary first step in implementing the block bootstrap method. To remove the linear trend, we estimated the magnitude of trend in a time series $\{X_i\}$ $(i = 1, \ldots, n)$ using

$$\hat{\beta} = Median(\frac{X_j - X_i}{j - i}), \forall i < j.$$

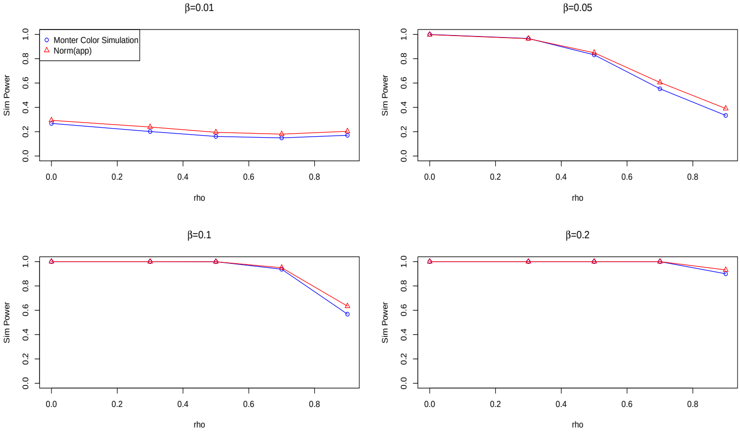
$\hat{\beta}$ is the estimate of the slope of the linear trend;
X_i is the i-th observation.

Power Comparison (Conti.)

We obtained the empirical power of *MC*, *Norm(app)* and *Bootprob* based on 10,000 tests for detecting positive linear, quadratic, and square root trends in the data with errors $\{\mu_t\}$ generated from normal AR(1) models.

$$X_t = \beta t + \mu_t$$
$$X_t = \beta t^2 + \mu_t$$
$$X_t = \beta \sqrt{t} + \mu_t$$

Empirical power of *MC* and *Normal(app)* for data with linear trend, normal AR(1) errors and $n = 50$ ($\alpha = 0.05$; $l = n^{1/3}$)

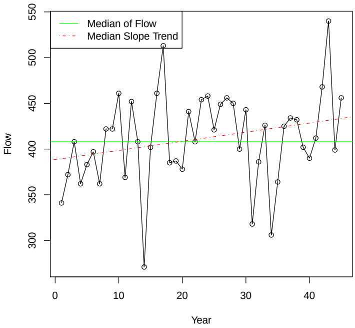


Power Comparison Results

- In the case of $\beta = 0$, the empirical significance levels for two bootstrap methods are very similar to the simulated values in the known null process, which is not the case for *MC*.
- The bootstrap method and the *MC* test have comparable powers for testing for trend.
- The power decreases with increasing ρ.
- The power for the square root case was found to be as good or better than in the linear case, whereas the opposite occurs for quadratic trend. In the latter case the power was still fairly good for $\beta = .05$, and for moderate values of ρ.

An Example

Stikine River Annual Mean Daily Streamflow (1965-2009)[5]



[5]Stikine river (BC) data were downloaded from the Reference Hydrometric Basin Network (RHBN) databases of Environment Canada (2010).

Example Results

Station identifier	Basin name	Record (yrs)	Mean (m^3/s)	Median ($m^3/s/yr$)	Median estimated slope $\hat{\beta}$	*Norm(app)* P-value	*IID* P-value
08CE001	Stikine river	45	410.98	408	1	0.054	0.030

Conclusions

- CLT holds for the Mann-Kendall statistic based on stationary ARMA processes.
- Using the Bootstrap method, the performance of the estimated null distribution of the Mann-Kendall tau statistic for stationary dependent data depends on the sample size, the model structure, and the block size.
- The *MC* needs complete information of the dependence structure, and without such knowledge the *MC* test is not applicable.

Conclusions (Conti.)

- In testing for trend in weakly dependent processes, with dependence structure and distribution information unknown, block bootstrap methods provide good approximations, when the sample size is relatively large or the autocorrelation is small.
- A practical approach to implementing the *Norm(app)* or *Bootprob* method can achieve reasonable empirical results.

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