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A comment on measuring the Hurst exponent of financial time series

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Abstract

A fundamental hypothesis of quantitative finance is that stock price variations are independent and can be modeled using Brownian motion. In recent years, it was proposed to use rescaled range analysis and its characteristic value, the Hurst exponent, to test for independence in financial time series. Theoretically, independent time series should be characterized by a Hurst exponent of 1/2. However, finite Brownian motion data sets will always give a value of the Hurst exponent larger than 1/2 and without an appropriate statistical test such a value can mistakenly be interpreted as evidence of long term memory. We obtain a more precise statistical significance test for the Hurst exponent and apply it to real financial data sets. Our empirical analysis shows no long-term memory in some financial returns, suggesting that Brownian motion cannot be rejected as a model for price dynamics.

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1. Introduction

A fundamental hypothesis of quantitative finance is that price variations are generated by a random process with no long-term memory. This hypothesis has been questioned starting with the work of Mandelbrot [1], who proposed that financial returns did indeed possess a long memory, which he and others suggest may be well described by fractional Brownian motion. Fractional Brownian motion is a long-memory modification of a standard Brownian motion process which is characterized by a so-called Hurst exponent $H \neq 1/2$:

$$\omega_H(t) = \frac{1}{\Gamma(H+1/2)} \int_0^t (t-\tau)^{H-1/2}\, \mathrm{d}\omega(\tau)\,,$$

where $\omega(t)$ is a standard Brownian motion. We see that when $H = 1/2$, fractional Brownian motion reduces to normal Brownian motion. However the work of Rogers [2] proves that fractional Brownian motion other than that corresponding to the trivial $H = 1/2$ case allows for arbitrage opportunities in which money can be made from financial markets without any risk being taken. This is anathema to financial theory. Were this possible, so the theory goes, people would do it and in so doing remove the statistical regularity which allowed the arbitrage in the first place. Even more recently, however, Øksendal [3] and his co-workers, as well as others, have suggested the use of the Wick products of quantum field theory as a technical way of avoiding the conclusion of Rogers. A certain debate is now raging in the quantitative finance community as to how to proceed—for an eloquent voice on the anti-Wick side of this debate see the recent preprint of Björk and Hult [4]. We prefer to avoid this debate entirely and to return to some of the statistical evidence for $H \neq 1/2$ in financial time series data. One popular way to measure H is rescaled range analysis. In this paper we show a naive application of this method can lead to erroneous identification of $H = 1/2$ data as long memory data. We obtain a revised statistical test to judge the significance of the Hurst exponent and apply our significance results to the returns of the S&P 500 index and of three companies, Procter & Gamble Corporation, General Electric Company and International Business Machines. At a 95% confidence level, we see that for each time series studied we cannot reject the null hypothesis of independence and therefore cannot reject the use of Brownian motion to model the return dynamics of the data sets studied.

2. Long-term memory in financial markets

The efficient market hypothesis (EMH) proposed by Fama [5] states that stock prices already reflect all available information useful in evaluating their value. This hypothesis has been widely criticized in the financial literature. The idea that all information is already included in the prices of assets would mean that a great deal of the research performed by portfolio managers and analysts will not generate higher returns and will only increase the administrative costs to be paid by their clients. We are interested in what is referred to as the weak form of the efficient market hypothesis. This form is a Markov assumption and states that all past price or volume fluctuations are all accounted for in the current price. Namely, it states that trend analysis, also known as technical analysis, cannot be used to gain additional information to be able to predict upward or downward movements in the stock price. This form of the EMH can easily be tested using historical data and studying the predictability in asset returns through short- and long-range correlations. It seems that the conclusion of most of the short-range correlation studies [6–8] is that the validity of the efficient market hypothesis tends to be verified. Although the weak form of the efficient market hypothesis seems to hold under short-time intervals, some long-range correlations studies [9,10] have demonstrated that large negative correlations exist over periods of many years and therefore are signs of the failure of the weak form of the hypothesis. These studies of long-range correlations were criticized by many and the debate over the validity of the weak form of the efficient market hypothesis is not yet resolved.

Empirical studies on the independence of financial asset returns have been published in the econophysics literature providing additional information on the validity or failure of the weak form of the efficient market hypothesis. Econophysics was created in recent years to describe the contributions of physicists to finance and economics [11,12]. Some statistical physicists study independence through the idea of scaling. In the case of independent normally distributed variables, the standard deviation behaves as $\sigma(t) \sim t^H$ where $H = 1/2$ and t is the time increment [13]. The exponent of this scaling relationship between the standard deviation of a time series and the time increments used is the Hurst exponent. Evidences of H different from $1/2$ could be interpreted as proof that returns are not independent and that long-term memory is present. Such memory would invalidate the weak form of the efficient market hypothesis.

Scaling in currency exchange was studied by Galluccio et al. [14]. The authors used data on the foreign exchange rates between the US Dollar, the German Mark and the Japanese Yen. An empirical exponent $H \approx 0.45$ was found between the standard deviation and the time increments. The authors stated that this exponent is not different from $1/2$ but that the underlying process is not independent as shown by the presence of autocorrelations. A second market, the futures market, was studied by Scalas [15]. The author found through the study of the autocorrelation function that no short-term correlations were present in the price fluctuations. The presence of long-range correlations was ruled out by finding a scaling exponent close to $1/2$, $H = 0.49$, between the volatility of the price changes and the time increments. Also, scaling behavior in stock prices in emerging markets was studied by Palágyi and Mantegna [16]. They found for four different stocks that the scaling exponents of the relationships between the standard deviation of the price changes and the time increments were slightly below $1/2$. The authors concluded that the fact that the exponents are below $1/2$ is proof of slight long-range correlations. However, we must note that none of these studies provide a statistical significance test to judge if the scaling exponents obtained are statistically different from the one characterizing an independent process and it may therefore be premature to conclude anything about the presence of long-range memory.

With similar scaling ideas, this time between the normalized range of accumulated departures from the mean and the length of the data observed, many studies have found evidences of long memory in asset returns. The technique used is known as rescaled range analysis. Greene and Fielitz [17] studied long-term dependence in common stock returns. Using the daily returns of 200 stocks listed on the New York Stock Exchange, they found that more than half of these stock returns time series were characterized by Hurst exponents larger than 1/2. The authors therefore concluded that long-term dependence in security returns was a "non-trivial problem". Booth et al. [18] studied foreign exchange rates using daily spot prices for the British Pound, the French Franc and the German Mark in terms of US Dollars under two different monetary policies. For a flexible rate policy, they found Hurst exponents larger than 1/2 and the authors claimed to have uncovered positive long term dependence. For a fixed rate policy, they found the situation was reversed and Hurst exponents smaller than 1/2 were signs of negative dependence. Also, Helms et al. [19] studied memory in commodity futures contracts. Using logarithmic differences in prices for six contracts related to soybeans, they found Hurst exponents ranging from 0.558 to 0.711 and concluded that persistence was present in these time series. More recently, Muniandy et al. [20] tested three different time series of foreign currency exchange rates with respect to the Malaysian currency. They obtained Hurst exponents between 0.54 and 0.62 and concluded for the presence of long-range dependence. Andreadis and Serletis [21] studied the behavior of the daily federal funds rate. Their rescaled range analysis produced a Hurst exponent of 0.93. Using this empirical value, they suggested that their dataset could be modeled using fractional Brownian motion.

We must point out that none of the studies we just mentioned include a precise significance test with which to judge the validity of their conclusions about the long-range effects in financial distributions. To be able to judge the real meaning of Hurst exponent values some studies tested the statistical significance of the Hurst exponent via simulations (see for example Bassingthwaighte and Raymond [22]). Peters [23] proposed a more precise statistical testing technique. He applied his significance results to the Hurst exponents obtained from the 20-day, 5-day and daily returns of the Dow Jones Industrial Average. He found that each time series had a Hurst exponent significantly different from 1/2 and was therefore characterized by persistence. Peters also found evidence of persistence in the S&P 500 returns, the Yen/Dollar exchange rates and the Mark/Dollar exchange rates. Using Peters' significance test, Skjeltorp [24] obtained signs of persistence in the daily changes of the prices of the Oslo Stock Exchange general index. Weron and Przybyowicz [25] also used Peters' technique in their analysis of electricity price dynamics. Using hourly spot prices from the California Power Exchange, they obtained Hurst exponents revealing the presence of antipersistence.

We studied Peters' statistical test and found it could be improved. Even if rescaled range analysis is to be used only for an exploratory look at data, knowing that the Hurst exponent obtained with this technique is biased, we still need a statistical test for this exploratory look to be appropriate.

3. Rescaled range analysis

Rescaled range analysis was created by Hurst [26] while studying the statistical properties of the Nile River overflows. This technique was inspired by Einstein's work on pure random walks, where the absolute value of the particles displacement scales as the square root of time. Hurst expressed the absolute displacement in terms of rescaled cumulative deviations from the mean (R/S_n) and defined time as the number of data points "n" used. The scaling exponent of the relationship $R/S_n =$ constant $\times n^H$, now referred to as the Hurst exponent, gives us information on the presence of long-range correlations in a time series. If the data set is purely independent, the distance traveled increases with the square root of time and the Hurst exponent will be 1/2. If antipersistence exists in a system, H will be smaller than 1/2 and if persistence occurs, H will be greater than 1/2.

By computing the values of the rescaled range R/S_n for different values of n, we can estimate the Hurst exponent H. We begin with a time series of length N and we divide it into M contiguous subperiods of length n such that $M \times n = N$. Each subperiod is labeled I_m with $m = 1, 2, \ldots, M$ and each element in I_m is labeled $N_{k,m}$ with $k = 1, 2, \ldots, n$. For each subperiod I_m, the average and the standard deviation are defined as

$$\mu_m = \frac{1}{n}\sum_{k=1}^{n} N_{k,m}, \quad S_m = \sqrt{\frac{1}{n}\sum_{k=1}^{n}(N_{k,m} - \mu_m)^2} .$$

The second step is to create a new time series of accumulated departures from the mean value for each subperiod I_m defined as

$$X_{k,m} = \sum_{i=1}^{k}(N_{i,m} - \mu_m) \quad k = 1, 2, \ldots, n .$$

Note that $X_{n,m} = 0$. We study the range of these accumulated departures from the mean

$$R_{I_m} = \max(X_{k,m}) - \min(X_{k,m}) \quad \text{where } 1 \leqslant k \leqslant n .$$

Each range is rescaled or normalized by dividing it by the standard deviation of its corresponding subperiod I_m. Finally, the rescaled range value for the length n is defined as

$$R/S_n = \frac{1}{M}\sum_{m=1}^{M} R_{I_m}/S_{I_m} .$$

The length n is then increased to the next higher value. We use only values of n that include the first and last points of the time series in order to always use the same number of points. Using $\log(R/S_n) = \log(c) + H\log(n)$, we run a regression to find the slope.

Mandelbrot [27–29] has exhaustively tested the possibility of using rescaled range analysis as an empirical technique to identify hidden cycles in data sets. This ability to identify cycle lengths comes from the definition of the rescaled range. Being based on the range of accumulated departures from the mean, the rescaled range cannot grow higher than the rescaled difference between the maximum and minimum values within a cycle. When cycles are present in a system, the rescaled range will grow linearly up to the point where n has the same value as the period of the underlying cycles. From this point, we will witness a departure from linear growth. By graphically identifying this departure value of n, we can identify the length of a cycle. It may not always be a precise measure of the period, but it gives insights on the behavior of the underlying process. A more fundamental observation is that when cycles are present in a system, we need to confine our analysis to values of n such that $n \leqslant n_{\max}$, where $n_{\max}$ is the period of the cycle.

Like any empirical technique, rescaled range analysis has shortcomings. The first of these shortcomings is the need for a large amount of data. It is well known that a Hurst exponent of 1/2 for an independent random process is an asymptotic limit. The use of time series with finite lengths will never yield the true value of the Hurst exponent [22]. Although data constraints in some fields can render rescaled range analysis useless, this is not the case in finance. The impressively large amount of financial data available from various databases enables us to reduce the imprecision related to the lack of data.

The second shortcoming of rescaled range analysis is the stationarity assumption. This test assumes that the underlying process remains the same throughout the sample. To test the validity of this assumption, the data set has to be divided in overlapping and non-overlapping subperiods. The Hurst exponents of each subperiod are then obtained and compared to the exponent of the entire sample to see if it is constant through time. Therefore, we are faced with a trade-off between data sufficiency and stationarity of the Hurst exponent. With a lot of data covering a long period of time, the accuracy of the Hurst exponent will be improved, but the stationarity of the exponent might become an issue.

The third and most fundamental shortcoming of rescaled range analysis is its sensitivity to short-range correlations. Davies and Harte [30] showed that short-range correlations can bias the Hurst exponent and lead to the conclusion that the process is exhibiting long-range memory. Two tools can be used to account for the bias produced by short-range correlations: the use of residuals or the modified R/S statistic. Using residuals from a single or multiple lag autoregressive model can remove the bias in the Hurst exponent. The drawback of this method lies in the choice of the appropriate number of lags to use. To avoid using residuals, we can use the modified R/S statistic proposed by Lo [31] in order to distinguish between long- and short-range dependence. The idea is to modify the rescaled range so that it is invariant over a general class of short memory processes. The proposed rescaled range is

$$R/S_n = \frac{1}{\sigma_a(q)}\left[\max_{1\leqslant k\leqslant n}\sum_{j=1}^{k}(X_j-\mu_a) - \min_{1\leqslant k\leqslant n}\sum_{j=1}^{k}(X_j-\mu_a)\right],$$

where

$$\sigma_a^2(q) \equiv \frac{1}{n}\sum_{j=1}^{n}(X_j-\mu_a)^2 + \frac{2}{n}\sum_{j=1}^{q} w_j(q)\left[\sum_{i=j+1}^{n}(X_i-\mu_a)(X_{i-j}-\mu_a)\right]$$

and

$$w_j(q) \equiv 1 - \frac{j}{q+1}, \quad q < n \, .$$

For $q = 0$, we get the original definition of the rescaled range. The modified standard deviation now takes into account the covariances of the first q lags. Lo states that by allowing q to increase with the number of observations n, but at a slower rate, R/S_n adjusts appropriately to general forms of short-range dependence. Long-range effects in a time series are identified using the asymptotic properties of a V_n statistic defined as $R/S_n/\sqrt{n}$. The modified rescaled range analysis no longer uses the slope of the regression between the logarithm of the rescaled range and the logarithm of the "window" size n.

The modified R/S statistic was used by many to study long-range memory in speculative assets [32–34]. However, Teverovsky et al. [35] made a thorough empirical investigation of the modified R/S statistic and found that the modification of the standard deviation using a lag q produced "serious drawbacks". They found that as the lag q increased, the modified R/S statistic had a strong bias towards accepting the null hypothesis of no long-range correlations. They argue that it is in practice difficult to choose an adequate value for the lag q. A value which is too small will not account for all short-range correlations, while a value which is too big will overcompensate and eliminate any long range effects existing in the time series. The appropriate value of q has to be derived from the underlying process which is usually unknown in real data sets. In view of Teverovsky et al. [35] work, we study the statistical significance of the original rescaled range analysis and its characteristic value, the Hurst exponent. Short range correlations in real data sets will be handled via the use of residuals.

Finally, we have to point out that Beran [36] showed that rescaled range analysis can give misleading estimates of the Hurst exponent in the presence of slowly decaying trends. However, short-term tests of the weak EMH showed there were no slowly decaying trends in real data sets, all short-range correlations did not persist for very long [6]. In light of these results, we will not account for the possibility of slowly decaying trends when estimating the value of the Hurst exponent with real financial data sets.

4. An empirical study of the statistical significance of the Hurst exponent

Our goal is to build a test in order to be able to accept or reject the following null hypothesis: the underlying process is Gaussian. The time series is an independent random walk. To achieve this goal we construct the following test statistic:

$$t = \frac{\text{Observed value of } H - \text{Mean of } H \text{ if independent}}{\text{Standard deviation of } H \text{ if independent}} \, .$$

4.1. Estimate of the mean value of the Hurst exponent for an independent process: $\hat{\mu}(H)$

Anis and Lloyd [37] derived the expected value of the rescaled range

$$E(R/S)_n = \frac{\Gamma((n-1)/2)}{\sqrt{\pi}\Gamma(n/2)} \sum_{r=1}^{n-1} \sqrt{\frac{n-r}{r}}\,.$$

Peters [23] argued that Anis and Lloyd's result overestimates the expected value of the rescaled range for small values of n. Peters proposed the following "empirical correction":

$$E(R/S)_n = \frac{n-1/2}{n} \sqrt{\frac{2}{n\pi}} \sum_{r=1}^{n-1} \sqrt{\frac{n-r}{r}}\,.$$

We decided to test the validity of Anis and Lloyd's result using simulations. Results are summarized in Table 1. We generated 10 000 times series of 5000 data points each. Each point was drawn from a normal distribution with zero mean and standard deviation one. Each time series is therefore independent and purely random. For each time series, we computed the rescaled range for various values of n. The average of the 10 000 values of each rescaled range value for a given n gives us an estimate of the expected value. We find that Anis and Lloyd's formula is accurate and can therefore be used to obtain an estimate of the mean value of the Hurst exponent for an independent process using a regression on the rescaled range values obtained with their formula. Anis and Lloyd's formula shows that the value $H = 1/2$

Table 1
Test on the validity of the result of Anis and Lloyd[a]

	Simulations	Formulas	
n	Average $\log(R/S_n)$	Anis and Lloyd	Peters
10	0.4805	0.4805	0.4233
20	0.6637	0.6638	0.6360
25	0.7207	0.7208	0.6986
40	0.8381	0.8382	0.8244
50	0.8926	0.8928	0.8819
100	1.0584	1.0589	1.0535
125	1.1109	1.1114	1.1070
200	1.2200	1.2207	1.1070
250	1.2712	1.2720	1.2698
500	1.4281	1.4298	1.4287
625	1.4784	1.4801	1.4792
1000	1.5820	1.5854	1.5849
1250	1.6334	1.6352	1.6348
2500	1.7835	1.7890	1.7888

[a]Simulations containing 5000 data points, 10 000 times.

for an independent process is an asymptotic limit and that for finite lengths the Hurst exponent will be biased high.

4.2. Obtaining $\hat{\mu}(H)$ to test real data sets

The estimate of the mean value of the Hurst exponent for an independent process may be obtained by two methods. We can simulate K independent time series, obtain K Hurst exponents from these time series and then average all values of the Hurst exponent. We can also use Anis and Lloyd's formula and fit a regression (which may be thought of as averaging first and then regressing). For $K \to \infty$, these two methods will give the same values. We have to point out that there are still two possible problems: we do not have an infinite number of simulations and Anis and Lloyd's formula might not be entirely accurate. To check, we compared our simulated values of $\hat{\mu}(H)$ to the values we would obtain using Anis and Lloyd's formula, for two lengths $N = 500$ and $N = 5000$. Table 2 shows that they are relatively close.

4.3. Estimate of the standard deviation of the Hurst exponent for an independent process: $\hat{\sigma}(H)$

Peters [23] argues that: "Because the R/S values are normally distributed random variables, the Hurst exponent is also normally distributed with (estimated) standard deviation: $\hat{\sigma}(H) = 1/\sqrt{N}$ where N is the total number of observations."

This may be true asymptotically, but we are not convinced that this argument holds for finite lengths. Therefore, we decided to study the behavior of the estimate of the standard deviation of the Hurst exponent for an independent process. We created time series of various lengths: $N = 500$, 1000, 2000, 3000, 4000, 5000, 8000, 10 000. Each data point was again drawn from a normal distribution with zero mean and standard deviation one to ensure that each time series was independent and purely random. For every length, 10 000 time series were created. From these 10 000 trials, we obtained 10 000 Hurst exponents. The standard deviation of these exponents is an estimate of the standard deviation of the Hurst exponent for an independent process. Results are summarized in Fig. 1 and Table 3. Our results for the expected standard deviation of the Hurst exponent for independent time series are shown with a 95% confidence interval.

We see that $\hat{\sigma}(H)$ does not behave like $1/\sqrt{N}$ for finite lengths. Our empirical values give higher standard deviations. A better approximation can be obtained

Table 2

N	Simulated $\hat{\mu}(H)$	$\hat{\mu}(H)$with Anis and Lloyd's formula
500	0.5614	0.5648
5000	0.5405	0.5419

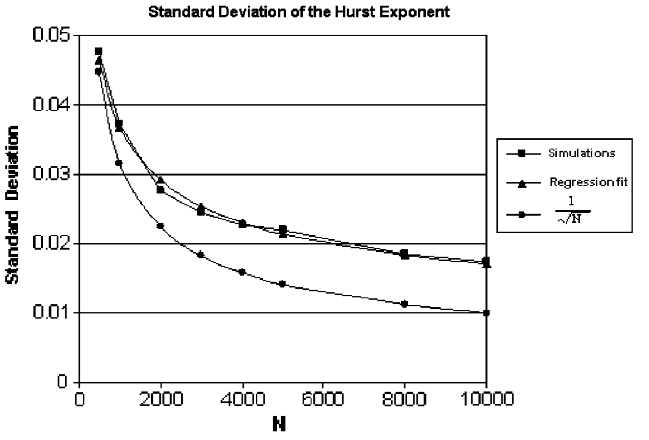

Fig. 1. Estimates of the standard deviation of the Hurst exponent for an independent process. The lower curve represents the values obtained using $1/\sqrt{N}$. The superposed curves show our simulation results and a linear regression fit of the logarithm of the standard deviation with respect to the logarithm of N.

Table 3
Estimate of the standard deviation of the Hurst exponent for an independent process

N	$\hat{\mu}(H)$	$\hat{\sigma}(H)$	Simulations 95% confidence interval for $\hat{\sigma}(H)$		$\frac{1}{\sqrt{N}}$	$\frac{1}{e\sqrt[3]{N}}$
			Lower bound	Upper bound		
500	0.5614	0.0477	0.0470	0.0483	0.0447	0.0463
1000	0.5544	0.0374	0.0368	0.0379	0.0316	0.0368
2000	0.5479	0.0275	0.0272	0.0279	0.0224	0.0292
3000	0.5455	0.0244	0.0241	0.0248	0.0183	0.0255
4000	0.5428	0.0226	0.0223	0.0229	0.0158	0.0232
5000	0.5405	0.0220	0.0217	0.0223	0.0141	0.0215
8000	0.5385	0.0184	0.0181	0.0186	0.0112	0.0184
10000	0.5369	0.0174	0.0172	0.0177	0.0100	0.0171

N = total number of observations.

by $\hat{\sigma}(H) \approx 1/eN^{1/3}$. This empirical relationship was obtained by running a regression between the logarithm of the standard deviation and the logarithm of the length of the time series. Results from this regression are given in Table 4. We see that the intercept could approximated by e^{-1} and that the slope is close to $-\frac{1}{3}$. Our results for the standard deviation show that it decreases at a slower rate with respect to N than it does with the square root proposed by Peters. Larger standard deviations can have an important impact on the statistical significance of the Hurst exponent.

Using Anis and Lloyd's formula to obtain $\hat{\mu}(H)$ and our simulation results to obtain $\hat{\sigma}(H)$, we can study real financial return data sets to see if they exhibit long-range memory.

Table 4
ln (σ) vs. ln (N)

Regression statistics					
R^2	0.9908				
Adjusted R^2	0.9893				
		Standard error	p-value (%)	Lower 95%	Upper 95%
Intercept	−0.9889	0.1057	0.008	−1.2477	−0.7302
Slope	−0.3351	0.0132	0.00002	−0.3674	−0.3029

4.4. *t-Statistic and p-value*

We now have the information we need to construct our test statistic in order to be able to judge the statistical significance of the Hurst exponents for real financial data sets. We can associate an exact level of significance, or p-value, with each of the values of our test statistic. To compute our p-values, we need to know the statistical distribution of our test statistic. Our observed Hurst exponents will be drawn from this distribution, so a histogram of these observed exponents will look like this distribution, especially for non-tail values. Our initial assumption is that the distribution of the Hurst exponents is single peaked with finite mean and variance. Through the averaging process of rescaled range analysis and with the use of a large number of observations, this distribution should be approximately normal. Fig. 2 shows the histogram of 10 000 Hurst exponents obtained using independent time series of 10 000 normally distributed observations. By comparing this distribution with a normal distribution having the same mean and variance, we see that our normality assumption seems realistic. Our test statistic therefore seems to be approximately normally distributed and we can use the normal distribution to obtain our p-values.

5. Financial data analysis

Using our revised statistical test for the significance of the Hurst exponent, we studied financial data sets. Our empirical study was based on the logarithmic returns of a stock index, the S&P 500, and of three large capitalization companies, Procter & Gamble Corporation, General Electric Company and International Business Machines. In the case of the S&P 500, we studied returns at three different frequencies, using daily, weekly and monthly returns. For the individual companies, we also studied daily, weekly and monthly returns, but we added intra-day returns using tick data. We used an intra-day frequency of 20 min.

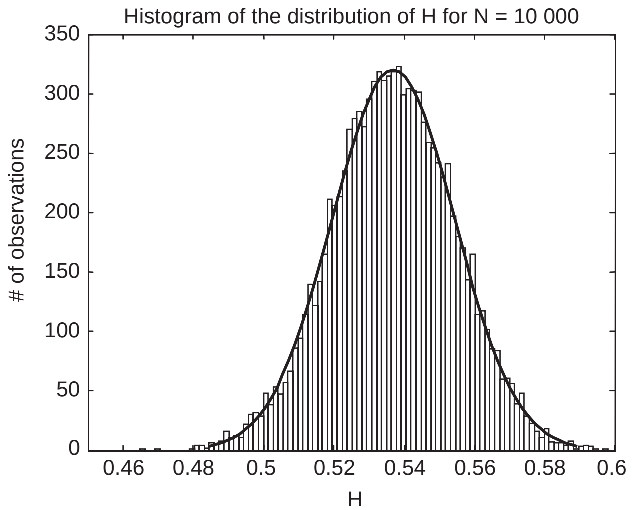

Fig. 2. Histogram of the distribution of 10 000 observed Hurst exponents from independent time series with length of 10 000 observations.

5.1. S&P 500

Our data set for the logarithmic daily returns of the S&P 500 index was composed of 7560 observations from June 20, 1973 to May 30, 2003. Our weekly data set included 2600 observations from July 27, 1953 to May 27, 2003. Finally, our monthly data set was composed of 600 observations from July 1, 1953 to June 2, 2003. Instead of studying returns, we used residuals obtained through an autoregressive process. To identify any short-range correlations, we first studied the autocorrelation up to 10 lags for each time series. We used a 95% confidence interval to select the significant lags. Then, we needed to identify any cycles in order to impose a maximum value of n before obtaining the value of the Hurst exponent. A cycle of 60 months seems to exist for monthly returns. Such a cycle seems realistic, corresponding to a 5 year business cycle. For daily and weekly returns however, no cycles are visible.

Results for the S&P 500 are shown in Table 5. At a 95% confidence interval, we see that we cannot reject the null hypothesis of independence for any of the three frequencies. No long-term memory is present. In order to study the stationarity of the Hurst exponent, each time series was divided into non-overlapping subperiods. We did not find any significant stationarity issues as our conclusions remained the same for the subperiods chosen. Therefore, our analysis shows that, based on a Hurst exponent test, Brownian motion cannot be rejected as a model for the daily and weekly return dynamics of the S&P 500. For the monthly returns case, we may need to introduce mean reversion to account for the cycles, but Brownian motion still cannot be rejected as a model of the random behavior of the time series.

Table 5

	Hurst	*t*-statistics	*p*-value(%)
S&P 500			
Daily	0.54	0.021	98
Weekly	0.57	0.79	43
Monthly	0.66	1.8	7.1

5.2. Procter & Gamble, General Electric and IBM

We conducted the same independence analysis on the logarithmic returns of three individual stocks. We selected three large capitalization companies traded on the New York Stock Exchange. The 20 minutes logarithmic returns data set included 800 observations for each company, from March 2001 to April 2001. The daily returns data set included 7560 observations for each company. The weekly returns data set and the monthly returns data set had respectively 1700 and 400 observations for each company.

It is important to note that we did not identify any cycle for any company at any frequency. Table 6 shows our results for each company. At a 95% confidence interval, we see that we cannot reject the null hypothesis of independence for the three companies at all four frequencies. Again, no long term memory is present. As in the S&P 500 case, the study of the stationarity of the Hurst exponent revealed no significant issues and our conclusions remained the same for the different subperiods chosen. Once again, our analysis shows that, based on a Hurst exponent test, Brownian motion cannot be rejected as a model for the return dynamics at different frequencies for Procter & Gamble Corporation, General Electric Company and IBM.

6. Conclusion

While the debate over the use of fractional Brownian motion and the Wick products is raging in the quantitative finance community, we prefer to return to some of the statistical evidence for long-term memory in financial time series. Rescaled range analysis and its characteristic value, the Hurst exponent, are used to test for independence in financial time series and therefore to test the validity of the weak form of the efficient market hypothesis. As finite data sets created using Brownian motion will always give a value of the Hurst exponent larger than 1/2, this value can therefore mistakenly be interpreted as evidence of long term memory and we need an appropriate statistical test to judge its significance.

We studied the use of rescaled range analysis and we obtained a revised statistical test to judge the significance of the Hurst exponent of a time series. Through simulations, we found that Anis and Lloyd's formula for the expected value of the

Table 6

	Hurst	t-statistics	p-value (%)
Procter & Gamble Corporation			
20 min	0.51	1.1	25
Daily	0.51	1.7	9.5
Weekly	0.55	0.094	93
Monthly	0.60	0.58	56
General Electric Company			
20 min	0.60	1.0	32
Daily	0.54	0.031	98
Weekly	0.57	0.46	65
Monthly	0.61	0.84	40
International Business Machines			
20 min	0.54	0.61	54
Daily	0.56	0.98	33
Weekly	0.59	1.2	22
Monthly	0.64	1.4	18

rescaled range was accurate and could therefore be used to obtain an estimate of the mean value of the Hurst exponent for an independent process. Then, we found a new empirical relation between the estimate of the standard deviation of the Hurst exponent for an independent process and the length of an independent time series: $\hat{\sigma}(H) \approx 1/eN^{1/3}$.

We applied our significance results to the logarithmic returns of the S&P 500, Procter & Gamble Corporation, General Electric Company and IBM. We studied different frequencies of returns, using 20 minutes, daily, weekly and monthly data. At a 95% confidence level, we saw that for each time series we could not reject the null hypothesis of independence. This analysis shows that using Hurst exponent tests, we cannot reject the use of Brownian motion to model the return dynamics of the data sets studied. Therefore the use of a more complex model like fractional Brownian motion might not be necessary.

An interesting question related to the one described in this paper is the following. Suppose we hypothesize that a given dataset is from a Hurst process with short effective memory, i.e., $1/2 - a < H < 1/2 + b$, where a and b are small positive numbers. Can a similar statistical test be devised to test this new hypothesis? In fact a simulation-based technique for hypothesis testing similar to the one described here could be devised. However, such a technique would need to contend with the difficulties in simulating fractional Brownian motion cited in [38]; this is especially true when $H < 1/2$. It would nonetheless be interesting to develop such a test to see what it has to say about datasets with measured values of H very different from $1/2$.

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