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Improved multivariate portmanteau test

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A new portmanteau diagnostic test for vector autoregressive moving average (VARMA) models that is based on the determinant of the standardized multivariate residual autocorrelations is derived. The new test statistic may be considered an extension of the univariate portmanteau test statistic suggested by Peña and Rodríguez (2002). The asymptotic distribution of the test statistic is derived as well as a chi-square approximation. However, the Monte–Carlo test is recommended unless the series is very long. Extensive simulation experiments demonstrate the usefulness of this test as well as its improved power performance compared to widely used previous multivariate portmanteau diagnostic check. Two illustrative applications are given.

Keywords: Diagnostic checking; multivariate time series; parallel computing; Monte Carlo significance test; residual autocorrelation function; VARMA models.

1. INTRODUCTION

The VARMA (p,q) model for a k-dimensional mean zero time series $\mathbf{Z}_t = (Z_{1,t}, \ldots, Z_{k,t})'$ can be written as

$$\boldsymbol{\Phi}(\boldsymbol{B})\boldsymbol{Z}_t = \boldsymbol{\Theta}(\boldsymbol{B})\boldsymbol{a}_t, \tag{1}$$

where $\boldsymbol{\Phi}(\boldsymbol{B}) = \mathbb{I}_k - \boldsymbol{\Phi}_1\boldsymbol{B} - \cdots - \boldsymbol{\Phi}_p\boldsymbol{B}^p, \boldsymbol{\Theta}(\boldsymbol{B}) = \mathbb{I}_k - \boldsymbol{\Theta}_1\boldsymbol{B} - \cdots - \boldsymbol{\Theta}_q\boldsymbol{B}^q$, $\mathbb{I}_k$ is the identity matrix of order k, the coefficient matricess are, $\boldsymbol{\Phi}_\ell = (\phi_{i,j,\ell})_{k\times k}, \ell = 1, \ldots, p$; $\boldsymbol{\Theta}_\ell = (\theta_{i,j,\ell})_{k\times k}$, $\ell = 1, \ldots, q$ and $\mathbf{B}$ is the backshift operator on t. Let $\boldsymbol{\beta} =$ (vec $\boldsymbol{\Phi}_1, \ldots,$ vec $\boldsymbol{\Phi}_p$, vec $\boldsymbol{\Theta}_1, \ldots,$ vec $\boldsymbol{\Theta}_q$) be the vector of true parameters, where vec denotes the matrix vectorization function. We assume that an efficient estimation algorithm such as maximum likelihood is used to produce the corresponding estimate $\hat{\boldsymbol{\beta}}$ so that $\hat{\boldsymbol{\beta}} - \boldsymbol{\beta} = O_p(n^{-1/2})$. The white noise process, $\boldsymbol{a}_t = (a_{1,t}, \ldots, a_{k,t})'$, is assumed independent normal with mean zero and covariance matrix, $E(\mathbf{a}_t\mathbf{a}_{t-\ell}') = \delta_\ell\boldsymbol{\Gamma}_0$, where $\boldsymbol{\Gamma}_0$ is the innovation covariance matrix and $\delta_\ell = 1$ or 0 according as $\ell = 0$ or $\ell \neq 0$. The assumption of normality may be relaxed to that of strong white noise so that $\boldsymbol{a}_t$, $t = 1, \ldots, n$ are assumed to be i.i.d. with mean zero and constant covariance matrix, $\boldsymbol{\Gamma}_0$. The model is assumed to be stationary, invertible and identifiable (Box *et al.*, 2008, § 14.2). After fitting this model to a series of length n, the residuals, $\hat{\boldsymbol{a}}_t = (\hat{a}_{1,t}, \ldots, \hat{a}_{k,t})'$, $t = 1, \ldots, n$ may be estimated and used to check the model assumption that the innovations are white noise, that is, to test the null hypothesis that

$$\mathcal{H}_0 : \boldsymbol{\Gamma}_\ell = 0, \quad \ell = 1, \ldots, m, \tag{2}$$

where $\boldsymbol{\Gamma}_\ell = \mathrm{Cov}\ \{\boldsymbol{a}_t, \boldsymbol{a}_{t-\ell}\}$ and m is chosen large enough to cover all lags, ℓ, of interest. Several versions of the multivariate portmanteau test have been developed for this purpose (Li, 2004).

In the next two subsections, brief reviews are given of previous multivariate portmanteau tests as well as the univariate versions of the generalized variance test of Peña and Rodríguez (2002, 2006). In Section 2, the multivariate extension of the generalized variance test of Peña and Rodríguez (2002) is discussed and its asymptotic distribution is derived. As in the univariate case (Peña and Rodríguez, 2002, eqn 9), it is shown in eqn (18) that the stronger the multivariate autocorrelation, the smaller the generalized variance. A chi-square approximation is suggested but for most purposes it is recommended to use a Monte–Carlo testing procedure that is described in Section 2.2. Simulation experiments in Section 3, demonstrate the improvement in power over the widely used previous multivariate portmanteau test. Illustrative applications are discussed in Section 4.

1.1. Multivariate portmanteau tests

The portmanteau test statistics, Q_m and $\tilde{Q}_m$ and others, discussed in this section are all asymptotically $\chi^2_{k^2(m-p-q)}$ as $n \to \infty$. It is also assumed that $m > p + q$ is fixed and that m large enough so that Theorem 5 in Li and McLeod (1981) holds.

Hosking (1980) defined the residual autocorrelation matrix,

$$\hat{\boldsymbol{R}}_\ell = \hat{\boldsymbol{L}}'\hat{\boldsymbol{\Gamma}}_\ell\hat{\boldsymbol{L}}, \tag{3}$$

where $\hat{\boldsymbol{\Gamma}}_\ell = n^{-1}\sum_{t=\ell+1}^{n}\hat{\boldsymbol{a}}_t\hat{\boldsymbol{a}}_{t-\ell}', \hat{\boldsymbol{\Gamma}}_{-\ell} = \hat{\boldsymbol{\Gamma}}_\ell', \ell \geq 0$ and $\hat{\boldsymbol{L}}$ is the lower triangular Cholesky decomposition of $\hat{\boldsymbol{\Gamma}}_0^{-1}$. The multivariate portmanteau test statistic may be written,

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$$Q_m = n \sum_{\ell=1}^{m} \hat{\boldsymbol{r}}_\ell{}' (\hat{\boldsymbol{R}}_0{}^{-1} \otimes \hat{\boldsymbol{R}}_0{}^{-1}) \hat{\boldsymbol{r}}_\ell, \tag{4}$$

where $\hat{\boldsymbol{r}}_\ell = \text{vec}\, \hat{\boldsymbol{R}}_\ell{}'$ is a row vector of length k^2 formed by stacking the rows of $\hat{\boldsymbol{R}}_\ell$, and m represents the number of lags being tested. In the univariate case, Q_m is identical to Box–Pierce portmanteau statistic [Box and Pierce, 1970] and both statistics are asymptotically $\chi^2_{k^2(m-p-q)}$ (Hosking, 1980, 1981b).

Li and McLeod (1981) defined,

$$\hat{\boldsymbol{R}}_\ell^{(\dagger)} = (\hat{r}_{i,j}(\ell))_{k \times k}, \tag{5}$$

where $\hat{r}_{i,j}(\ell) = \hat{\gamma}_{i,j}(\ell)/\sqrt{(\hat{\gamma}_{i,i}(0)\hat{\gamma}_{j,j}(0))}$, $i, j = 1, \ldots, k$ and $\hat{\gamma}_{i,j}(\ell) = n^{-1} \sum_{t=\ell+1}^{n} \hat{a}_{i,t}\hat{a}_{j,t-\ell}$, $\hat{\gamma}_{i,j}(-\ell) = \hat{\gamma}_{j,i}(\ell), \ell \geq 0$. Replacing $\hat{\boldsymbol{R}}$ by $\hat{\boldsymbol{R}}^{(\dagger)}$ in eqn (4), another portmanteau test statistic $Q_m^{(\dagger)}$ is obtained. The null distribution of $Q_m^{(\dagger)}$ is also asymptotically $\chi^2_{k^2(m-p-q)}$. The definition of residual autocorrelations used in eqn (3) is equivalent to the residual autocorrelations in eqn (5) if the residuals used eqn (5), $\hat{\boldsymbol{a}}_t$, are replaced by the standardized residuals, $\hat{\boldsymbol{L}}'\hat{\boldsymbol{a}}_t$.

Chitturi (1974) defined the residual autocorrelation matrix at lag ℓ,

$$\hat{\boldsymbol{R}}_\ell^{(\ddagger)} = \hat{\boldsymbol{\Gamma}}_\ell \hat{\boldsymbol{\Gamma}}_0{}^{-1}, \tag{6}$$

and another portmanteau test statistic $Q_m^{(\ddagger)}$ is obtained by replacing $\hat{\boldsymbol{R}}$ by $\hat{\boldsymbol{R}}^{(\ddagger)}$ in eqn (4), and its null distribution is also asymptotically $\chi^2_{k^2(m-p-q)}$.

Hosking (1981b) noted that $Q_m = Q_m^{(\dagger)} = Q_m^{(\ddagger)}$ and the portmanteau test statistic may be expressed simply in terms of the residual autocovariances,

$$Q_m = n \sum_{\ell=1}^{m} \text{tr}\, (\hat{\boldsymbol{\Gamma}}_\ell{}' \hat{\boldsymbol{\Gamma}}_0{}^{-1} \hat{\boldsymbol{\Gamma}}_\ell \hat{\boldsymbol{\Gamma}}_0{}^{-1}), \tag{7}$$

where tr (•) denotes trace of matrix. The multivariate portmanteau test statistic is equivalent to a test based on the Lagrange multiplier (Hosking, 1981a; Poskitt and Tremayne, 1982).

Hosking (1980) and Li and McLeod (1981) suggested modified versions of Q_m so that the expected value of the modified portmanteau statistic under the null hypothesis is equal to $k^2(m - p - q) + O_p(1/n)$ and showed that both of these modifications are satisfactory when n and m are large enough. Simulation experiments suggest that both these modified portmanteau tests work about equally well (Li, 2004, §3).

The modified portmanteau test of Hosking (1980) is given by,

$$\tilde{Q}_m = n^2 \sum_{\ell=1}^{m} \hat{\boldsymbol{r}}_\ell' (\hat{\boldsymbol{R}}_0{}^{-1} \otimes \hat{\boldsymbol{R}}_0{}^{-1}) \hat{\boldsymbol{r}}_\ell / (n - \ell). \tag{8}$$

In the univariate time series, the $\tilde{Q}_m$ test statistic approximately equal the Ljung–Box statistic (Ljung and Box, 1978) and both statistics are asymptotically $\chi^2_{k^2(m-p-q)}$ (Hosking, 1980, 1981b).

1.2. Univariate generalized variance portmanteau test

Peňa and Rodríguez (2002) proposed a univariate portmanteau test statistic,

$$\hat{D}_m = n(1 - |\hat{\mathcal{R}}_m|^{1/m}), \tag{9}$$

where |•| denotes the determinant and $\hat{\mathcal{R}}_m$ is the residual correlation matrix of order $m + 1$,

$$\hat{\mathcal{R}}_m = \begin{pmatrix} 1 & \hat{r}_{11}(1) & \ldots & \hat{r}_{11}(m) \\ \hat{r}_{11}(1) & 1 & \ldots & \hat{r}_{11}(m-1) \\ \vdots & \ldots & \ddots & \vdots \\ \hat{r}_{11}(m) & \hat{r}_{11}(m-1) & \cdots & 1 \end{pmatrix}. \tag{10}$$

Peňa and Rodríguez (2002) derived the asymptotic distribution of $\hat{D}_m$ as gamma using the standardized values of residual autocorrelations. Li (2004, §2.7) noted several interesting interpretations for this statistic. It was shown in simulation experiments (Peňa and Rodríguez 2002) that the $\hat{D}_m$ statistic had better power than the test of Ljung and Box (1978) in many situations. One problem noted by Lin and McLeod (2006) is that the test statistic $\hat{D}_m$ may not exist because, with the modified version of the residual autocorrelations used, the residual autocorrelation sequence is not always positive-definite or even non-negative definite.

Furthermore, the size of the test may not be accurate due to the asymptotic approximation (Li, 2004, p. 19). To overcome these difficulties Lin and McLeod (2006) suggested using a Monte–Carlo significance test and demonstrated that this approach provides a test with the correct size and is often more powerful than the usual Ljung–Box test (Lin and McLeod, 2006, Table 6).

Peňa and Rodríguez (2006) suggested taking the log of the $(m + 1)$th root of the determinant in eqn (10),

$$\tilde{D}_m = -n(m+1)^{-1} \log |\hat{\mathcal{R}}_m| \tag{11}$$

and they derived a gamma distribution approximation for this test statistic.

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In the portmanteau tests based on the asymptotic distribution (Ljung and Box, 1978; Peňa and Rodríguez, 2002, 2006) not only is the size of the test inaccurate if the series length n is not large enough but there is also a problem if m, the number of lags, is not large enough as well. The Monte–Carlo significance test approach does not require any such assumption about m and has much better finite-sample properties than tests based on the asymptotic distribution.

2. NEW MULTIVARIATE PORTMANTEAU TEST

The univariate residual autocorrelations in the Toeplitz matrix in eqn (10) are replaced by, $\hat{\boldsymbol{R}}_\ell, \ell = 1, \ldots, m$ in eqn (3),

$$\hat{\mathfrak{R}}_m = \begin{pmatrix} \mathbb{I}_k & \hat{\boldsymbol{R}}_1 & \ldots & \hat{\boldsymbol{R}}_m \\ \hat{\boldsymbol{R}}_1' & \mathbb{I}_k & \ldots & \hat{\boldsymbol{R}}_{m-1} \\ \vdots & \ldots & \ddots & \vdots \\ \hat{\boldsymbol{R}}_m' & \hat{\boldsymbol{R}}_{m-1}' & \ldots & \mathbb{I}_k \end{pmatrix}, \tag{12}$$

where $\mathbb{I}_k = \hat{\boldsymbol{R}}_0$. The proposed multivariate portmanteau test statistic is

$$\mathfrak{D}_m = -n \log |\hat{\mathfrak{R}}_m|. \tag{13}$$

From Hadamard's inequality for the determinant of a positive definite matrix, $|\hat{\mathfrak{R}}_m| \leq 1$. When there is no significant autocorrelation in the residuals, $\hat{\boldsymbol{R}}_\ell = O_p(n^{-\frac{1}{2}})$ so $\hat{\mathfrak{R}}_m$ is approximately block diagonal and hence $|\hat{\mathfrak{R}}_m| \approx 1$.

On the other hand, when there is autocorrelation present, $|\hat{\mathfrak{R}}_m|$ will be expected to be smaller than 1. To see this we repeatedly apply the formula for the determinant of a partitioned matrix [Seber, 2008, §14.1],

$$|\hat{\mathfrak{R}}_m| = \prod_{\ell=1}^{m} |\mathbb{I}_k - \hat{\mathfrak{R}}_{(\ell)} \hat{\mathfrak{R}}_{\ell-1}^{-1} \hat{\mathfrak{R}}_{(\ell)}'|, \tag{14}$$

where $\hat{\mathfrak{R}}_{(\ell)} = [\hat{\boldsymbol{R}}_1 : \cdots : \hat{\boldsymbol{R}}_\ell]$ is the k-by-ℓk block partitioned matrix. Then $\hat{\Sigma}_\ell = \mathbb{I}_k - \hat{\mathfrak{R}}_{(\ell)} \hat{\mathfrak{R}}_{\ell-1}^{-1} \hat{\mathfrak{R}}_{(\ell)}'$ corresponds to the error covariance matrix when a linear predictor of order ℓ is fit to $\hat{\boldsymbol{L}}' \hat{\boldsymbol{a}}_t$ using the previous ℓ values (Reinsel, 1997, eqn 3.15). Thus, eqn (14) is a direct multivariate generalization of the well known univariate decomposition of generalized variance into the product of the one-step ahead variances of the linear minimum-mean-square error predictors (McLeod, 1977, p. 532),

$$|\hat{\mathcal{R}}_m| = \prod_{\ell=1}^{m} \hat{\sigma}_\ell^2, \tag{15}$$

where $\hat{\sigma}_\ell^2$ is the mean-square error for a fitted linear predictor of order ℓ. In this case, $R_\ell^2 = 1 - \hat{\sigma}_\ell^2$, where R_ℓ^2 is the square of the multiple correlation for the order ℓ linear predictor, and so (Peňa and Rodríguez, 2002, eqn 7),

$$|\hat{\mathcal{R}}_m| = \prod_{\ell=1}^{m} (1 - R_\ell^2). \tag{16}$$

In the multivariate case,

$$\hat{\eta}_\ell^2 = 1 - |\mathbb{I}_k - \hat{\mathfrak{R}}_{(\ell)} \hat{\mathfrak{R}}_{\ell-1}^{-1} \hat{\mathfrak{R}}_{(\ell)}'| \tag{17}$$

is the proportion of the generalized variance that is accounted for by a linear predictor of order ℓ. From eqns (14) and (17), the corresponding multivariate equivalent of eqn (16) is

$$|\hat{\mathfrak{R}}_m| = \prod_{\ell=1}^{m} (1 - \hat{\eta}_\ell^2). \tag{18}$$

It follows from eqn (18), $|\hat{\mathfrak{R}}_m| < 1$ and that the smaller the value of $|\hat{\mathfrak{R}}_m|$, the more strongly autocorrelated the normalized residuals, $\hat{\boldsymbol{L}}' \hat{\boldsymbol{a}}_t$, are.

Using the Chitturi (1974) multivariate residual autocorrelations, eqn (6), the correlation matrix corresponding to eqn (12), $\hat{\mathfrak{R}}_m^{(\ddagger)}$, is defined by the block matrix with (i, j)-block, $\hat{\boldsymbol{R}}_{i-j}^{(\ddagger)}$ for $i, j = 1, \ldots, m + 1$. This matrix is not symmetric but $|\hat{\mathfrak{R}}_m| = |\hat{\mathfrak{R}}_m^{(\ddagger)}|$, so these multivariate autocorrelations could also be used.

Multivariate autocorrelations are often defined as in eqn (5) (Box *et al.*, 2008, eqn 14.1.2). Using this definition, the residual autocorrelation matrix may be written,

$$\hat{\boldsymbol{R}}_\ell^{(\dagger)} = \hat{\boldsymbol{D}}^{-1/2} \hat{\Gamma}_\ell \hat{\boldsymbol{D}}^{-1/2}, \tag{19}$$

where $\hat{\boldsymbol{D}}^{-1/2} = \text{diag}\,(\hat{\gamma}_{1,1}^{-1/2}(0), \ldots, \hat{\gamma}_{k,k}^{-1/2}(0))$. The correlation matrix corresponding to eqn (12) obtained by replacing $\hat{\boldsymbol{R}}_\ell$ by $\hat{\boldsymbol{R}}_\ell^{(\dagger)}$ may be denoted by $\hat{\mathfrak{R}}_\ell^{(\dagger)}$ and the corresponding generalized variance portmanteau statistic, $|\hat{\mathfrak{R}}_m^{(\dagger)}|$. A similar decomposition as given in eqn (18) shows that small values $|\hat{\mathfrak{R}}_m^{(\dagger)}|$ correspond to positive autocorrelation. On the other hand, when there is no autocorrelation

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present, the off-block diagonal entries in the matrix $\hat{\mathfrak{R}}_m^{(\dagger)}$ are $O_p(n^{-1/2})$. So, $|\hat{\mathfrak{R}}_m^{(\dagger)}| \approx |\hat{\boldsymbol{R}}_0^{(\dagger)}|^{m+1}$. When the innovation variance matrix, $\boldsymbol{\Gamma}_0$, has large off-diagonal elements, $|\hat{\boldsymbol{R}}_0^{(\dagger)}| < 1$. Hence again $|\hat{\mathfrak{R}}_m^{(\dagger)}| = O_p(r^m)$ for some $r \in (0, 1)$. So, in both cases, autocorrelation or no autocorrelation, $|\hat{\mathfrak{R}}_m^{(\dagger)}|$ tends to be small provided the innovation covariance matrix is not diagonal. Numerical experiments confirmed that the test using $\mathfrak{D}_m^{(\dagger)}$ and $\mathfrak{D}_m$ is essentially equivalent when $\boldsymbol{\Gamma}_0$ is diagonal but in the non-diagonal case, $\mathfrak{D}_m^{(\dagger)}$ does not provide a useful test.

2.1. Asymptotic distribution and approximation

Here, the asymptotic distribution for $\mathfrak{D}_m$ in eqn (13) is derived and an approximation to this distribution is suggested. Since, as shown in Lin and McLeod (2006, Figure 2) in the univariate case by simulation, the actual finite-sample distribution for $\mathfrak{D}_m$ converges slowly, the asymptotic distribution for $\mathfrak{D}_m$ is not expected to be of much use in diagnostic checking multivariate time series models unless n is very large.

We use the following notation as in Hosking (1980, §4), $\boldsymbol{\Psi}(\boldsymbol{B}) = \boldsymbol{\Phi}(\boldsymbol{B})^{-1}\boldsymbol{\Theta}(\boldsymbol{B}) = \sum_{i=0}^{\infty} \Psi_i B^i$ and $\boldsymbol{\Pi}(\boldsymbol{B}) = \boldsymbol{\Theta}(\boldsymbol{B})^{-1} = \sum_{i=0}^{\infty} \Pi_i B^i$ are matrix power series such that the elements Ψ_i and Π_i converge exponentially to zero as $i \to \infty$. Define

$$\boldsymbol{G} = \begin{pmatrix} G_0 & 0 & \dots & 0 \\ G_1 & G_0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ G_{m-1} & G_{m-2} & \cdots & G_{m-p} \end{pmatrix}, \tag{20}$$

and

$$\boldsymbol{H} = \begin{pmatrix} H_0 & 0 & \dots & 0 \\ H_1 & H_0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H_{m-1} & H_{m-2} & \cdots & H_{m-q} \end{pmatrix}, \tag{21}$$

where $G_r = \sum_{i=0}^{\infty} \boldsymbol{\Gamma}_0 \Psi_i' \otimes \Pi_{r-i}$ and $H_r = \boldsymbol{\Gamma}_0 \otimes \Pi_r$.

THEOREM 1. *Assume that the model specified in eqn (1) has i.i.d. innovations with mean zero and constant covariance matrix. The model is fit to a series of length n using an $n^{-1/2}$-consistent algorithm. After obtaining the residuals defined in eqn (3) and the test statistic, $\mathfrak{D}_m$, in eqn (13), $\mathfrak{D}_m$ is asymptotically distributed as*

$$\sum_{i=1}^{k^2 m} \lambda_i \chi^2_{1,i},$$

where $\chi^2_{1,i}, i = 1, \dots, k^2 m$ are independent χ^2_1 random variables and $\lambda_1, \dots, \lambda_{k^2 m}$ are the eigenvalues of $(\mathbb{I}_{k^2} - \boldsymbol{Q})\boldsymbol{M}$, where $\boldsymbol{M}$ is $k^2 m \times k^2 m$ diagonal matrix

$$\boldsymbol{M} = \begin{pmatrix} m\mathbb{I}_{k^2} & \boldsymbol{O} & \dots & \boldsymbol{O} \\ \boldsymbol{O} & (m-1)\mathbb{I}_{k^2} & \dots & \boldsymbol{O} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{O} & \boldsymbol{O} & \cdots & \mathbb{I}_{k^2} \end{pmatrix}, \tag{22}$$

and

$$\boldsymbol{Q} = \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{W}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{W}^{-1} \tag{23}$$

is an idempotent matrix with rank $k^2(p+q)$, $\boldsymbol{X}$ is defined as $k^2 m \times k^2(p+q)$ matrix **(G−H)**, *and $\boldsymbol{W} = \mathbb{I}_m \otimes \boldsymbol{\Gamma}_0 \otimes \boldsymbol{\Gamma}_0$ is positive-definite symmetric.*

PROOF. From the decomposition in eqn (14), it follows that,

$$-n \log |\hat{\mathfrak{R}}_m| = -n \sum_{\ell=1}^{m} \log |\mathbb{I}_k - \boldsymbol{A}_\ell|, \tag{24}$$

where $\boldsymbol{A}_\ell = \hat{\mathfrak{R}}_{(\ell)} \hat{\mathfrak{R}}_{\ell-1}^{-1} \hat{\mathfrak{R}}_{(\ell)}'$. Using the fact that $|\mathbb{I}_k - \boldsymbol{A}_\ell| = \prod_{i=1}^{k}(1 - \lambda_i(\ell))$, where $\lambda_i(\ell)$ are the eigenvalues of $\boldsymbol{A}_\ell$, $\ell = 1, \dots, m$,

$$-n \log |\hat{\mathfrak{R}}_m| = -n \sum_{\ell=1}^{m} \sum_{i=1}^{k} \log(1 - \lambda_i(\ell)). \tag{25}$$

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Expanding $\log(1 - \lambda_i(\ell)) = -\sum_{r=1}^{\infty} r^{-1}\lambda_i^r(\ell)$ and $\mathrm{tr}\,(\boldsymbol{A}_\ell) = \sum_{i=1}^{k}\lambda_i(\ell)$,

$$\mathfrak{D}_m = n\sum_{\ell=1}^{m}\mathrm{tr}\,(\boldsymbol{A}_\ell) + O_p(n^{-1}). \tag{26}$$

One can verify that

$$\begin{aligned}\mathrm{tr}\,(\boldsymbol{A}_1) &= \mathrm{tr}\,(\hat{\boldsymbol{R}}_1'\hat{\boldsymbol{R}}_1)\\ \mathrm{tr}\,(\boldsymbol{A}_2) &\approx \mathrm{tr}\,(\hat{\boldsymbol{R}}_1'\hat{\boldsymbol{R}}_1) + \mathrm{tr}\,(\hat{\boldsymbol{R}}_2'\hat{\boldsymbol{R}}_2)\\ &\vdots\\ \mathrm{tr}\,(\boldsymbol{A}_m) &\approx \mathrm{tr}\,(\hat{\boldsymbol{R}}_1'\hat{\boldsymbol{R}}_1) + \cdots + \mathrm{tr}\,(\hat{\boldsymbol{R}}_m'\hat{\boldsymbol{R}}_m),\end{aligned} \tag{27}$$

so that,

$$\mathfrak{D}_m \approx n\sum_{\ell=1}^{m}(m-\ell+1)\,\mathrm{tr}\,(\hat{\boldsymbol{R}}_\ell'\hat{\boldsymbol{R}}_\ell). \tag{28}$$

Using the commutative property of trace,

$$\mathfrak{D}_m \approx n\sum_{\ell=1}^{m}(m-\ell+1)\,\mathrm{tr}\,(\hat{\boldsymbol{\Gamma}}_\ell{}'\hat{\boldsymbol{\Gamma}}_0{}^{-1}\hat{\boldsymbol{\Gamma}}_\ell\hat{\boldsymbol{\Gamma}}_0{}^{-1}). \tag{29}$$

It follows from Neudecker (1969, eqn 2.12),

$$\begin{aligned}\mathfrak{D}_m &\approx n\sum_{\ell=1}^{m}(m-\ell+1)(\mathrm{vec}\,\hat{\boldsymbol{\Gamma}}_\ell)'(\hat{\boldsymbol{\Gamma}}_0^{-1}\otimes\hat{\boldsymbol{\Gamma}}_0^{-1})\mathrm{vec}\,\hat{\boldsymbol{\Gamma}}_\ell,\\ &= n(\mathrm{vec}\,\hat{\boldsymbol{\Gamma}})'(\mathbb{I}_m\otimes\hat{\boldsymbol{\Gamma}}_0^{-1}\otimes\hat{\boldsymbol{\Gamma}}_0^{-1})\boldsymbol{M}(\mathrm{vec}\,\hat{\boldsymbol{\Gamma}}),\end{aligned} \tag{30}$$

where $\mathrm{vec}\,\hat{\boldsymbol{\Gamma}} = (\mathrm{vec}\,\hat{\boldsymbol{\Gamma}}_1 \ldots \mathrm{vec}\,\hat{\boldsymbol{\Gamma}}_m)$ is $k^2m \times 1$ column vector and $\boldsymbol{M}$ is $k^2m \times k^2m$ diagonal matrix defined in eqn (22).

Hosking [1980, Theorem 1] showed that

$$\sqrt{n}\,\mathrm{vec}\,\hat{\boldsymbol{\Gamma}} \sim N_{k^2m}(\boldsymbol{0}, (\mathbb{I}_{k^2m} - \boldsymbol{Q})\boldsymbol{W}), \tag{31}$$

where $\boldsymbol{W}^{-1}$ can be replaced by a consistent estimator $\hat{\boldsymbol{W}}^{-1} = \mathbb{I}_m\otimes\hat{\boldsymbol{\Gamma}}_0^{-1}\otimes\hat{\boldsymbol{\Gamma}}_0^{-1}$, and $\boldsymbol{Q}$ is the idempotent matrix with rank $k^2(p+q)$ in eqn (23).

From the theorem on quadratic forms given by Box (1954, Theorem 2.1), and eqns (30) and (31), the asymptotic distribution of $\mathfrak{D}_m$ is given by,

$$\mathfrak{D}_m \to \sum_{i=1}^{k^2m}\lambda_i\chi_1^2, \tag{32}$$

where $\to$ stands for convergence in distribution as $n\to\infty$ and $\lambda_1, \ldots, \lambda_{k^2m}$ are the eigenvalues of $(\mathbb{I}_{k^2m} - \boldsymbol{Q})\boldsymbol{M}$. □

2.1.1. Approximation

The upper percentiles of the cumulative distribution function in eqn (32) could be evaluated by the Imhof (1961) algorithm. For the univariate case, Lin and McLeod (2006, Table 2) showed that the convergence to the asymptotic distribution is very slow. In the case of large-samples, an approximation based on Box (1954, Theorem 3.1) works well. Using this result, the test statistic in eqn (32) can be approximated by $a\chi_b^2$, where a and b are chosen to make the first two moments agree with those of exact distribution of $\mathfrak{D}_m$. Hence, $a = \sum\lambda_i^2/\sum\lambda_i$ and $b = (\sum\lambda_i)^2/\sum\lambda_i^2$, where,

$$\begin{aligned}\sum_{i=1}^{k^2m}\lambda_i &= \mathrm{tr}\,(\mathbb{I}_{k^2m} - \boldsymbol{Q})\boldsymbol{M},\\ \sum_{i=1}^{k^2m}\lambda_i^2 &= \mathrm{tr}\,(\mathbb{I}_{k^2m} - \boldsymbol{Q})\boldsymbol{M}(\mathbb{I}_{k^2m} - \boldsymbol{Q})\boldsymbol{M}.\end{aligned} \tag{33}$$

When $p = q = 0$, $a = (2m+1)/3$ and $b = 1.5k^2m(m+1)/(2m+1)$. In the VARMA (p, q) case, one degree of freedom is lost for each parameter so $\mathfrak{D}_m$ is approximately distributed as $a\chi_b^2$, where

$$\begin{aligned}a &= \frac{2m+1}{3},\\ b &= \frac{3k^2m(m+1)}{2(2m+1)} - k^2(p+q).\end{aligned} \tag{34}$$

2.2. Monte–Carlo significance test

Monte–Carlo significance tests, originally suggested by Barnard (1963), are feasible for many small-sample problems (Marriott, 1979) and with modern computing facilities these types of tests are increasingly feasible for larger samples and more complex problems

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(Dufour and Khalaf, 2001). For a pure significance test with no nuisance parameters, as is the case, for example, for simply testing a time series for randomness, accuracy of the Monte–Carlo procedure depends only on the number of simulations (Dufour, 2006, Proposition 2.1).

In the case of diagnostic checking, the model parameters must be estimated and Dufour (2006, Proposition 5.1) has shown that, provided consistent estimators are used, Monte–Carlo tests remain asymptotically valid. Since we assume $n^{-1/2}$-consistent estimators are used, the requirements for Dufour (2006, Proposition 5.1) are met.

Simulations for $\mathfrak{D}_m$ in the univariate case (Lin and McLeod, 2006 Table 3) as well as our simulations for the multivariate case in Section 3.1, suggest the impact of nuisance parameters is negligible. The *p*-value for all of the portmanteau test statistics presented in this article may be obtained using the Monte–Carlo method outlined below. We use the statistic $\mathfrak{D}_m$ in the description but $\tilde{Q}_m$ could be used instead.

Step 1: Set *N*, the number of simulations. Usually, $N \leftarrow 1000$ but smaller values may be used if necessary. By choosing *N* large enough, an accurate estimate of the *p*-value may be obtained.

Step 2: After fitting the model and obtaining the residuals, compute the portmanteau test statistic for lag *m* or possibly a set of lags such as $\ell = 1, \ldots, m$, where $m \geq 1$. Typically *m* is chosen large enough to allow for possible high-order autocorrelations. Denote the observed value of the test statistics by $\mathfrak{D}_\ell^{(o)}, \ell = 1, \ldots, m$.

Step 3: For each $i = 1, \ldots, N$, simulate the fitted model, refit it, obtain the residuals from this model, compute the test statistic, $\mathfrak{D}_\ell^{(i)}, \ell = 1, \ldots, m$.

Step 4: For each ℓ, $\ell = 1, \ldots, m$, the estimated *p*-value is given by,

$$\hat{\text{p}} = \frac{\#\{\mathfrak{D}_\ell^{(i)} \geq \mathfrak{D}_\ell^{(o)}, i = 1, 2, \ldots, N\} + 1}{N + 1}. \tag{35}$$

The approximate 95% margin of error for the *p*-value is, $1.96\sqrt{\hat{\text{p}}(1 - \hat{\text{p}})/N}$.

The above algorithm is simply a restatement of the Monte–Carlo testing algorithm given by Lin and McLeod (2006, §3) for the univariate case. Lin and McLeod (2006, Table 3) demonstrate that the Monte–Carlo testing procedure has the correct size for an AR (1) and this is verified for some VAR (1) models in Section 3.1.

Remark 1. In the Monte–Carlo test procedure, it is assumed that the innovations used in our simulations in Step 3 are normally distributed but any distribution with constant covariance matrix could be used. In particular, using the empirical joint distribution is equivalent to bootstrapping the multivariate residuals. Using bootstrapped residuals is implemented in our software (Mahdi and McLeod, 2011).

Remark 2. A limitation of the Monte–Carlo diagnostic check is the assumption of constant variance. Many financial time series exhibit conditional heteroscedasticity. In practice this means that our test may overstate the significance level (Duchesne and Lalancette, 2003). This means that when used for constructing a VAR or VARMA model, the final fitted model may not be as parsimonious as a model developed using a portmanteau test which takes into conditional heteroscedasticity (Francq and Raïsi, 2007; Duchesne, 2006). Our Monte–Carlo portmanteau test can also be used to test for the presence of multivariate conditional heteroscedasticity simply by replacing the residuals by squared or absolute residuals. An illustration of this procedure is given later in Section 4.2.

Remark 3. Francq and Raïsi (2007) discuss a more general asymptotic multivariate portmanteau diagnostic test that is valid assuming only that the innovations are uncorrelated. This test requires a large sample though.

Remark 4. Lin and McLeod (2008) discuss the Monte–Carlo portmanteau test for univariate ARMA with infinite variance. The Monte–Carlo method of Lin and McLeod (2008) for infinite-variance ARMA has been extended to the multivariate case as well and is available in our R package (Mahdi and McLeod, 2011).

3. SIMULATION RESULTS

The purpose of our simulations is to demonstrate the improved power as well as the correct size of the Monte–Carlo (MC) test using $\mathfrak{D}_m$. We also compare the empirical Type 1 error rates for the $a\chi_b^2$-approximation discussed in Section 2.1.1.

3.1. Comparison of type 1 error rates

The empirical error rates have been evaluated under the Gaussian bivariate VAR (1) process $\boldsymbol{Z}_t = \boldsymbol{\Phi}_i \boldsymbol{Z}_{t-1} + \boldsymbol{a}_t, i = 1, \ldots, 4$ for the portmanteau test statistic $\mathfrak{D}_m$ using the MC and $a\chi_b^2$-approximation to evaluate the *p*-value. The covariance matrix of $\boldsymbol{a}_t$ has unit variances and covariance 1/2 and the coefficient matrices are taken from Hosking (1980) and Li and McLeod (1981),

$$\boldsymbol{\Phi}_1 = \begin{pmatrix} 0.9 & 0.1 \\ -0.6 & 0.4 \end{pmatrix}, \boldsymbol{\Phi}_2 = \begin{pmatrix} -1.5 & 1.2 \\ -0.9 & 0.5 \end{pmatrix}, \boldsymbol{\Phi}_3 = \begin{pmatrix} 0.4 & 0.1 \\ -1.0 & 0.5 \end{pmatrix}, \boldsymbol{\Phi}_4 = \begin{pmatrix} 0.3 & 0.5 \\ 0.0 & 0.3 \end{pmatrix}.$$

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Table 1. The empirical 5% significance level, in percent, comparing approximation, $a\chi_b^2$, and Monte–Carlo, MC, for the portmanteau test statistic $\mathfrak{D}_m$

		$n=100$		$n=200$		$n=500$	
	m	$a\chi_b^2$	MC	$a\chi_b^2$	MC	$a\chi_b^2$	MC
$\mathbf{\Phi}_1$							
	5	5.9	4.6	5.1	4.7	4.8	4.8
	10	5.2	4.5	4.4	5.2	3.7	4.2
	15	5.7	5.4	4.5	4.4	3.6	3.8
	20	6.8	5.8	4.8	4.0	3.8	3.8
	25	7.8	4.9	5.3	4.1	4.0	4.0
	30	9.0	4.8	5.8	3.7	4.4	4.1
$\mathbf{\Phi}_2$							
	5	4.7	4.8	4.0	4.8	3.5	4.7
	10	4.8	3.8	3.8	4.0	3.5	4.8
	15	5.7	3.9	4.3	3.9	3.6	5.0
	20	6.9	4.2	4.9	4.2	3.8	4.8
	25	8.2	4.0	5.3	3.9	4.1	5.3
	30	9.5	4.3	5.8	4.0	4.5	5.4
$\mathbf{\Phi}_3$							
	5	4.0	4.6	3.6	5.7	3.2	5.2
	10	4.5	4.8	3.8	6.5	3.1	5.3
	15	5.1	4.2	4.1	6.3	3.3	5.1
	20	6.6	4.3	4.6	6.2	3.6	5.2
	25	7.7	4.5	5.3	5.4	4.0	5.3
	30	9.0	4.2	5.9	5.5	4.3	5.0
$\mathbf{\Phi}_4$							
	5	2.9	4.3	2.6	4.7	2.5	5.2
	10	3.9	4.6	3.2	4.9	3.0	4.5
	15	4.9	4.1	3.9	4.6	3.2	5.0
	20	6.1	4.4	4.5	5.3	3.6	4.9
	25	7.3	3.9	5.0	5.0	3.9	4.8
	30	8.7	3.9	5.6	5.2	4.3	4.7

The empirical error rates are shown in Table 1. For each entry in Table 1, 10^3 simulations were done. The MC test also used $N = 10^3$.

The 95% confidence interval assuming the 5% rejection rate for each test is (3.6,6.4). There are 17 entries outside this interval with the $a\chi_b^2$ approximation and only one with the Monte–Carlo test. In conclusion, size-distortion with the Monte–Carlo test appears to be negligible but is sometimes present when the $a\chi_b^2$ approximation is used.

In Section 4, we found that there is a much larger discrepancy between the p-values using the $a\chi_b^2$ approximation and those using the Monte–Carlo test.

3.2. Power comparisons

Only Monte–Carlo significance tests are used to compare the empirical power of 5% level tests with $\tilde{Q}_m$ and $\mathfrak{D}_m$. Possible size-distortion sometimes makes power comparisons between asymptotic tests and Monte–Carlo tests invalid. In our comparisons, VAR models are fitted to various multivariate models. The power of diagnostic tests using $\mathfrak{D}_m$ versus $\tilde{Q}_m$ are compared using simulation. In all comparisons, the p-values were evaluated using the Monte–Carlo (MC) method with $N = 10^3$. We consider a VAR (1) model fitted to simulated data generated from eight VARMA models selected from well-known textbooks as cited below.

Model 1

Lütkepohl (2005, p. 17).

$$\begin{bmatrix} Z_{1,t} \\ Z_{2,t} \end{bmatrix} - \begin{bmatrix} 0.5 & 0.1 \\ 0.4 & 0.5 \end{bmatrix} \begin{bmatrix} Z_{1,t-1} \\ Z_{2,t-1} \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0.3 & 0 \end{bmatrix} \begin{bmatrix} Z_{1,t-2} \\ Z_{2,t-2} \end{bmatrix} = \begin{bmatrix} a_{1,t} \\ a_{2,t} \end{bmatrix}$$

$$\mathbf{\Gamma}_0 = \begin{pmatrix} 1.00 & 0.71 \\ 0.71 & 1.00 \end{pmatrix}$$

Model 2

Brockwell and Davis (1991, p. 428).

$$\begin{bmatrix} Z_{1,t} \\ Z_{2,t} \end{bmatrix} - \begin{bmatrix} 0.7 & 0 \\ 0 & 0.6 \end{bmatrix} \begin{bmatrix} Z_{1,t-1} \\ Z_{2,t-1} \end{bmatrix} = \begin{bmatrix} a_{1,t} \\ a_{2,t} \end{bmatrix} - \begin{bmatrix} 0.5 & 0.6 \\ -0.7 & 0.8 \end{bmatrix} \begin{bmatrix} a_{1,t-1} \\ a_{2,t-1} \end{bmatrix}$$

$$\mathbf{\Gamma}_0 = \begin{pmatrix} 1.00 & 0.71 \\ 0.71 & 2.00 \end{pmatrix}$$

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Table 2. Empirical power comparison of $\mathfrak{D}_m$ and $\tilde{Q}_m$ for a nominal 5% test

		$n = 50$		$n = 100$		$n = 200$	
Model	m	$\mathfrak{D}_m$	$\tilde{Q}_m$	$\mathfrak{D}_m$	$\tilde{Q}_m$	$\mathfrak{D}_m$	$\tilde{Q}_m$
1	5	35	24	68	53	96	90
1	10	24	16	55	36	90	73
1	15	18	14	46	30	85	61
1	20	13	13	39	26	80	52
1	30	10	12	30	23	68	43
2	5	70	48	100	94	100	100
2	10	60	38	99	82	100	100
2	15	50	35	99	75	100	100
2	20	43	34	97	70	100	99
2	30	28	37	93	64	100	97
3	5	99	84	100	100	100	100
3	10	96	64	100	99	100	100
3	15	93	48	100	97	100	100
3	20	88	39	100	91	100	100
3	30	73	36	100	77	100	100
4	5	51	27	93	62	100	98
4	10	37	24	84	48	100	89
4	15	27	22	74	40	99	81
4	20	20	22	65	37	98	73
4	30	13	22	53	33	95	65
5	5	99	68	100	100	100	100
5	10	95	46	100	93	100	100
5	15	90	36	100	81	100	100
5	20	83	32	100	72	100	100
5	30	69	30	100	60	100	97
6	5	83	45	100	90	100	100
6	10	74	32	100	69	100	100
6	15	62	28	99	57	100	96
6	20	54	28	98	52	100	92
6	30	40	27	95	44	100	84
7	5	29	21	65	49	97	91
7	10	19	14	53	33	92	74
7	15	14	12	43	27	86	61
7	20	13	11	35	22	82	53
7	30	11	11	27	19	72	41
8	5	77	28	96	85	100	100
8	10	65	19	92	61	100	99
8	15	52	17	84	48	100	94
8	20	38	14	76	40	100	90
8	30	15	13	55	33	100	78

Power is in percent. 10^4 simulations with $N = 10^3$.

Table 3. IBM and S&P 500 Index Data

	VAR(1)				VAR(3)				VAR(5)			
	$a\chi^2_b$		MC		$a\chi^2_b$		MC		$a\chi^2_b$		MC	
m	$\mathfrak{D}_m$	$\tilde{Q}_m$	$\mathfrak{D}_m$	$\tilde{Q}_m$	$\mathfrak{D}_m$	$\tilde{Q}_m$	$\mathfrak{D}_m$	$\tilde{Q}_m$	$\mathfrak{D}_m$	$\tilde{Q}_m$	$\mathfrak{D}_m$	$\tilde{Q}_m$
5	0.2	*	*	*	10.4	0.6	1.7	0.6	NA	NA	91.2	89.9
10	0.1	0.3	*	0.2	13.5	6.1	2.8	4.0	77.4	50.3	59.4	50.2
15	0.3	2.1	*	2.2	20.4	22.3	6.4	22.1	84.0	61.2	63.1	61.4
20	0.2	*	*	*	15.4	2.6	5.0	2.2	71.8	11.3	45.2	9.9
25	0.1	*	*	*	8.7	1.1	2.3	0.7	53.0	7.6	27.5	7.1
30	0.2	*	*	*	7.3	2.7	2.3	2.2	46.2	13.7	23.3	12.0

$a\chi^2_b$, approximation. MC, Monte–Carlo $N = 10^3$. NA, not applicable. The p-values are in percent. The * indicates a p-value less than 0.1%.

Model 3

Reinsel (1997, p. 81).

$$\begin{bmatrix} Z_{1,t} \\ Z_{2,t} \end{bmatrix} - \begin{bmatrix} 1.2 & -0.5 \\ 0.6 & 0.3 \end{bmatrix} \begin{bmatrix} Z_{1,t-1} \\ Z_{2,t-1} \end{bmatrix} = \begin{bmatrix} a_{1,t} \\ a_{2,t} \end{bmatrix} - \begin{bmatrix} -0.6 & 0.3 \\ 0.3 & 0.6 \end{bmatrix} \begin{bmatrix} a_{1,t-1} \\ a_{2,t-1} \end{bmatrix}$$

$$\Gamma_0 = \begin{pmatrix} 1.00 & 0.50 \\ 0.50 & 1.25 \end{pmatrix}$$

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Model 4

Tsay (2005, 2nd edn, p. 371).

$$\begin{bmatrix} Z_{1,t} \\ Z_{2,t} \end{bmatrix} - \begin{bmatrix} 0.8 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Z_{1,t-1} \\ Z_{2,t-1} \end{bmatrix} = \begin{bmatrix} a_{1,t} \\ a_{2,t} \end{bmatrix} - \begin{bmatrix} -0.5 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_{1,t-1} \\ a_{2,t-1} \end{bmatrix}$$

$$\boldsymbol{\Gamma}_0 = \begin{pmatrix} 1.00 & 0.71 \\ 0.71 & 1.00 \end{pmatrix}$$

Model 5

Reinsel (1997, p. 25).

$$\begin{bmatrix} Z_{1,t} \\ Z_{2,t} \end{bmatrix} = \begin{bmatrix} a_{1,t} \\ a_{2,t} \end{bmatrix} - \begin{bmatrix} 0.8 & 0.7 \\ -0.4 & 0.6 \end{bmatrix} \begin{bmatrix} a_{1,t-1} \\ a_{2,t-1} \end{bmatrix}$$

$$\boldsymbol{\Gamma}_0 = \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}$$

Model 6

Tsay (2005, 2nd edn, p. 350).

$$\begin{bmatrix} Z_{1,t} \\ Z_{2,t} \end{bmatrix} = \begin{bmatrix} a_{1,t} \\ a_{2,t} \end{bmatrix} - \begin{bmatrix} 0.2 & 0.3 \\ -0.6 & 1.1 \end{bmatrix} \begin{bmatrix} a_{1,t-1} \\ a_{2,t-1} \end{bmatrix}$$

$$\boldsymbol{\Gamma}_0 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Model 7

Lütkepohl (2005, p. 445).

$$\begin{bmatrix} Z_{1,t} \\ Z_{2,t} \end{bmatrix} - \begin{bmatrix} 0.5 & 0.1 \\ 0.4 & 0.5 \end{bmatrix} \begin{bmatrix} Z_{1,t-1} \\ Z_{2,t-1} \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0.25 & 0 \end{bmatrix} \begin{bmatrix} Z_{1,t-2} \\ Z_{2,t-2} \end{bmatrix} = \begin{bmatrix} a_{1,t} \\ a_{2,t} \end{bmatrix} - \begin{bmatrix} 0.6 & 0.2 \\ 0 & 0.3 \end{bmatrix} \begin{bmatrix} a_{1,t-1} \\ a_{2,t-1} \end{bmatrix}$$

$$\boldsymbol{\Gamma}_0 = \begin{pmatrix} 1.0 & 0.3 \\ 0.3 & 1.0 \end{pmatrix}$$

Model 8

Reinsel *et al.* (1992, p. 141).

$$\begin{bmatrix} Z_{1,t} \\ Z_{2,t} \\ Z_{3,t} \end{bmatrix} - \begin{bmatrix} 0.4 & 0.3 & -0.6 \\ 0.0 & 0.8 & 0.4 \\ 0.3 & 0.0 & 0.0 \end{bmatrix} \begin{bmatrix} Z_{1,t-1} \\ Z_{2,t-1} \\ Z_{3,t-1} \end{bmatrix} = \begin{bmatrix} a_{1,t} \\ a_{2,t} \\ a_{3,t} \end{bmatrix} - \begin{bmatrix} 0.7 & 0.0 & 0.0 \\ 0.1 & 0.2 & 0.0 \\ -0.4 & 0.5 & -0.1 \end{bmatrix} \begin{bmatrix} a_{1,t-1} \\ a_{2,t-1} \\ a_{3,t-1} \end{bmatrix}$$

$$\boldsymbol{\Gamma}_0 = \begin{pmatrix} 1.0 & 0.5 & 0.4 \\ 0.5 & 1.0 & 0.7 \\ 0.4 & 0.7 & 1.0 \end{pmatrix}$$

The power of the portmanteau statistics $\mathfrak{D}_m$ and $\tilde{Q}_m$ for nominal 5% tests using the MC test are shown in Table 2. The power is evaluated for 10^4 simulations for each parameter setting and and $N = 10^3$ is used in the MC algorithm. It is clear from Table 2 that the $\mathfrak{D}_m$ test is often substantially more powerful than the $\tilde{Q}_m$. Only when $n = 50$ and $m = 30$ is the $\tilde{Q}_m$ test more powerful and this only occurs for Models 2 and 4.

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Table 4. Trivariate West German macroeconomic series

	VAR(0)				VAR(1)				VAR(2)			
	$a\chi^2_b$		MC		$a\chi^2_b$		MC		$a\chi^2_b$		MC	
m	$\mathfrak{D}_m$	$\tilde{Q}_m$	$\mathfrak{D}_m$	$\tilde{Q}_m$	$\mathfrak{D}_m$	$\tilde{Q}_m$	$\mathfrak{D}_m$	$\tilde{Q}_m$	$\mathfrak{D}_m$	$\tilde{Q}_m$	$\mathfrak{D}_m$	$\tilde{Q}_m$
5	*	*	0.1	0.1	3.1	4.7	2.2	4.8	33.1	29.8	31.2	38.0
10	*	0.6	0.3	0.5	4.0	14.7	7.0	12.7	49.5	48.0	54.2	50.6
15	*	0.2	0.4	0.6	4.1	13.7	17.7	12.4	32.8	34.6	56.2	35.5

$a\chi^2_b$, approximation. MC, Monte–Carlo using 10^3 replications. The p-values are in percent and * indicates a p-value less than 0.1%.

Table 5. The residuals of the fitted VAR(2) model on West German macroeconomic series are tested for heteroscedastic effects

			MC	
m	$\mathfrak{D}_m$	$\tilde{Q}_m$	$\mathfrak{D}_m$	$\tilde{Q}_m$
5	0.2	15.2	31.9	81.3
10	0.3	6.3	24.4	37.9
15	*	*	12.2	1.6

$a\chi^2_b$, approximation. MC, Monte-Carlo using 10^3 replications. The p-values are in percent and * indicates a p-value less than 0.1%.

4. ILLUSTRATIVE APPLICATIONS

4.1. IBM and S&P index

Tsay (2010, chap. 8) uses the portmanteau diagnostic test in constructing a VAR model for the monthly log returns of IBM stock and the S&P 500 index for January 1926 to December 2008. So here, $n = 996$. Univariate analysis for both of these series indicates the presence of conditional heteroscedasticity (Tsay, 2010, p. 408) but for forecasting purposes, we may consider a VAR model rather a more complex VAR/GARCH model (Weiss, 1984; Francq and Raïsi, 2007). There are $n = 996$ and the AIC selects a VAR(5) model. We found that the BIC selects a VAR(1) model. Table 3 compares the p-values for the portmanteau tests for the VAR(p) for $p = 1,3,5$.

These portmanteau tests suggest that the VAR(5) is adequate and that the VAR(1) and VAR(3) both exhibit lack of fit. The VAR(4) is not shown but the results for this model are similar to the VAR(3). As noted in Remark 2, the presence of conditional heteroscedasticity means that the p-values in Table 3 are too small and this implies that, possibly, a lower-order model than the VAR(5) may be adequate. This possibility could be investigated using the multivariate portmanteau test of Francq and Raïsi (2007).

Table 3 also shows that $a\chi^2_b$ approximation for the p-value of $\mathfrak{D}_m$ is inaccurate whereas for $\tilde{Q}_m$ the asymptotic approximation agrees quite well with the Monte–Carlo result.

4.2. Investment, income and consumption time series

The trivariate quarterly time series, 1960–1982, of West German investment, income and consumption was discussed by Lütkepohl (2005, §3.2.3). For this series, $n = 92$ and $k = 3$. As in Lütkepohl (2005, §4.3.1) we model the logarithms of the first differences. Using the AIC, Lütkepohl (2005, Table 4.5) selected a VAR (2) for this data. Only lags $m = 5,10,15$ are used in the diagnostic checks since n is relatively short. All diagnostic tests reject simple randomness, VAR (0). The Monte–Carlo tests for VAR (1) suggests model inadequacy at lag 5. Table 4 supports the choice of the VAR (2) model.

As pointed out in Remark 2, we may test for multivariate heteroscedasticity by using the squared residuals and Table 5 gives the p-values with this test for the VAR(2) model. In this case, $a\chi^2_b$ approximation for $\mathfrak{D}_m$ as well as the asymptotic χ^2 approximation for $\tilde{Q}_m$ are quite inaccurate. Based on the Monte–Carlo tests there is little evidence to reject that null hypothesis of constant variance.

5. CONCLUDING REMARKS

Box *et al.* (2008) stress the importance of constructing an adequate and parsimonious model in which the residuals pass a suitable portmanteau diagnostic check. In forecasting experiments with monthly riverflow time series, Noakes *et al.* (1985) found that simply using a criterion such as the AIC or BIC may provide a model that either does not pass a suitable diagnostic check for randomness of the residuals or that may have more parameters than necessary. Monthly riverflow time series models chosen with the fewest number of parameters that pass the portmanteau diagnostic check for periodic autocorrelation (McLeod, 1994) tend to produce better one-step ahead forecasts (Noakes *et al.*, 1985). McLeod (1993) suggested formulating the principle of parsimony as an optimization problem: minimize model complexity subject to model adequacy. In any case, in the overall approach suggested many years ago and presented in their recent book (Box *et al.*, 2008), portmanteau diagnostic checks play a crucial role in constructing time series models.

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In Section 2.2, Remark 2, it was pointed out the Monte-Carlo test with $\mathfrak{D}_m$ may also be useful in diagnostic checking for multivariate conditional heteroscedasticity when used with squared or absolute residuals. This test is implemented in Mahdi and McLeod (2011). There is an extensive literature on testing residuals in VAR and VARMA models for conditional heteroscedasticity (Ling and Li, 1997; Dunchesne and Lalancette, 2003; Duchesne 2004; Rodríguez and Ruiz, 2005; Duchesne, 2006; Chabot-Hall and Duchesne, 2008). The power study presented Section 3.2 suggests that the $\mathfrak{D}_m$ with squared or absolute residuals may be useful. Peňa and Rodríguez (2002) also suggested that using squared-residuals with their generalized-variance portmanteau test would outperform the usual diagnostic check (McLeod and Li, 1983). Other tests designed for particular alternatives might be expected to perform better than an omnibus portmanteau test such as $\mathfrak{D}_m$ or $\tilde{Q}_m$ when these alternatives hold. For example, Rodríguez and Ruiz (2005) developed a diagnostic check for heteroscedasticity for the case of small autocorrelations.

The multivariate portmanteau diagnostic test developed by Francq and Raïsi (2007) does not require independent and identically innovations but only uncorrelated innovations. This test would be appropriate for the bivariate example in Section 4.1.

Scripts for reproducing all tables in this article are available with our freely available software (Mahdi and McLeod, 2011). This package can utilize multicore CPUs often found in modern personal computers as well as a computer cluster or grid (Schmidberger *et al.*, 2009). On a modern eight core personal computer, the computations for Tables 4 and 5 take about one minute. Table 3 takes about six minutes due to the longer series length and increased number of lags. The simulations reported in Section 3 were run on a computer cluster.

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