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Derivation of the Theoretical Autocovariance Function of Autoregressive–Moving Average Time Series

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SUMMARY

An algorithm suitable for the machine calculation of the theoretical autocovariance function is suggested.

Keywords: AUTOREGRESSIVE-MOVING AVERAGE TIME SERIES (ARMA); SEASONAL ARMA; THEORETICAL AUTOCOVARIANCE; TIME SERIES SIMULATION

1. INTRODUCTION

EXACT methods of calculating the theoretical autocovariance function for autoregressive models are given by Quenouille (1947) and Pagano (1973). The procedure to be described is used by Box and Jenkins (1970) to derive formulae for the theoretical autocovariance of some simple ARMA and seasonal ARMA models although the general algorithm is not explicitly stated. This method is exact, easily coded and can be used for ARMA and seasonal ARMA models of all orders.

Consider the stationary ARMA (p,q) model:

$$z_t-\phi_1 z_{t-1}-\ldots-\phi_p z_{t-p} = a_t-\theta_1 a_{t-1}-\ldots-\theta_q a_{t-q}, \tag{1}$$

$E(a_t)=0$, $E(a_t^2)=\sigma_a^2$ and $E(a_t a_s)=0$, $t\neq s$. It is noted by Box and Jenkins (1970, p. 74) that

$$\gamma_k-\phi_1\gamma_{k-1}-\ldots-\phi_p\gamma_{k-p} = \gamma_{za}(k)-\theta_1\gamma_{za}(k-1)-\ldots-\theta_q\gamma_{za}(k-q), \tag{2}$$

where $\gamma_k = E(z_{t-k}z_t)$ and $\gamma_{za}(k) = E(z_{t-k}a_t)$. Multiplying equation (1) by a_{t-k} and taking expectations we obtain

$$\gamma_{za}(-k)-\phi_1\gamma_{za}(-k+1)-\ldots-\phi_p\gamma_{za}(-k+p) = -[\theta_k]\,\sigma_a^2, \tag{3}$$

where

$$[\theta_k] = \begin{cases} \theta_k, & k=1,\ldots,q,\\ -1, & k=0,\\ 0, & \text{otherwise};\end{cases}$$

and $\gamma_{za}(k)=0$ if $k>0$ (Box and Jenkins, 1970, p. 75). If $k>r=\max(p,q)$, equation (2) may be used to calculate γ_k directly from previous values. The algorithm below is obtained by solving equations (2) and (3) for γ_k, $k=0,\ldots,r$.

2. Algorithm

(i) Set $\phi_0 = \theta_0 = -1$, $c_0 = 1$ and

$$c_k = -\theta_k + \sum_{i=1}^{\mathrm{Min}(p,k)} \phi_i c_{k-i}$$

for $k = 1, \ldots, q$.

(ii) Set

$$b_k = \sum_{i=k}^{q} \theta_i c_{i-k}$$

for $k = 0, \ldots, q$ and $b_k = 0$ if $k > q$.

(iii) If $p = 0$, $\gamma_k = b_k \sigma_a^2$, $k = 0, \ldots, q$. If $p > 0$, solve the equations $Ax = y$ where

$$A_{ij} = \begin{cases} [\phi_{i-1}], & j = 1; \quad i = 1, \ldots, r+1, \\ [\phi_{i-j}] + [\phi_{i+j-2}], & j = 2, \ldots, r+1; \quad i = 1, \ldots, r+1, \end{cases}$$

$$[\phi_k] = \begin{cases} \phi_k, & k = 0, 1, \ldots, p, \\ 0, & \text{otherwise} \end{cases}$$

$$y_i = -b_{i-1} \sigma_a^2$$

and then set $\gamma_k = x_{k+1}$, $k = 0, \ldots, r$.

Note that $c_k = \gamma_{za}(-k)/\sigma_a^2 = \psi_k$ where ψ_k is the coefficient of a_{t-k} in the infinite moving average representation of the model (Box and Jenkins, 1970, p. 46) and b_k is the right-hand side of equation (2).

3. Remarks

This procedure is useful in simulating initial values of a time series in simulation studies and calculating the large sample variances and covariances of the sample autocorrelations. If an algebraic processor, such as ALTRAN, is available, specific formulae are easily obtained. For example, the seasonal autoregression,

$$(1 - \phi_1 B)(1 - \Phi_1 B^4) z_t = a_t,$$

has theoretical autocorrelation function, $\rho_k = \gamma_k/\gamma_0$, determined by

$$\rho_1 = \phi_1(1 + \phi_1^2 \Phi_1)/D, \quad \rho_2 = \phi_1^2(1 + \Phi_1)/D,$$

$$\rho_3 = \phi_1(\Phi_1 + \phi_1^2)/D \quad \text{and} \quad \rho_4 = (\Phi_1 + \phi_1^4)/D,$$

where $D = 1 + \phi_1^4 \Phi_1$.

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Correction

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Step (ii) of the algorithm on p. 256 of my paper should read as follows:

(ii) Set

$$b_k = -\sum_{i=k}^{q} \theta_i c_{i-k}$$

for $k = 0, \ldots, q$ and $b_k = 0$ if $k > q$.