HYPERBOLIC DECAY TIME SERIES

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Abstract

Hyperbolic decay time series such as, fractional Gaussian noise (FGN) or fractional autoregressive moving-average (FARMA) process, each exhibit two distinct types of behaviour: strong persistence or antipersistence. Beran (1994) characterized the family of strongly persistent time series. A more general family of hyperbolic decay time series is introduced and its basic properties are characterized in terms of the autocovariance and spectral density functions. The random shock and inverted form representations are derived. It is shown that every strongly persistent series is the dual of an antipersistent series and vice versa. The asymptotic generalized variance of hyperbolic decay time series with unit innovation variance is shown to be infinite which implies that the variance of the minimum mean-square error one-step linear predictor using the last k observations decays slowly to the innovation variance as k gets large.

Keywords. Covariance determinant; duality in time series; fractional differencing and fractional Gaussian noise; long-range dependence; minimum mean square error predictor; nonstationary time series modelling.

1. INTRODUCTION

Let Z_t, $t = 1, 2, \ldots$ denote a covariance stationary, purely nondeterministic time series with mean zero and with autocovariance function, $\gamma_Z(k) = \operatorname{cov}(Z_t, Z_{t-k})$. As is discussed by Beran (1994), many long memory processes such as the FGN (Mandelbrot, 1983) and FARMA (Granger and Joyeux, 1980; Hosking, 1981) may be characterized by the property that $k^\alpha \gamma_Z(k) \to c_{\alpha,\gamma}$ as $k \to \infty$, for some $\alpha \in (0,1)$ and $c_{\alpha,\gamma} > 0$. Equivalently,

$$\gamma_Z(k) \sim c_{\alpha,\gamma}\, k^{-\alpha}. \tag{1}$$

As noted in Box and Jenkins (1976), the usual stationary ARMA models on the other hand are exponentially damped since $\gamma_Z(k) = O(r^k)$, $r \in (0,1)$.

Beran (1994, p.42) shows that an equivalent characterization of strongly persistent time series is

$$f_Z(\lambda) \sim c_f \lambda^{\alpha-1} \quad \text{as} \quad \lambda \to 0, \tag{2}$$

where $\alpha \in (0,1)$, $c_f > 0$ and $f_Z(\lambda)$ is the spectral density function given by $f_Z(\lambda) = \sum \gamma_Z(k) e^{-ik\lambda}/(2\pi)$. Theorem 1 below summarizes some results stated without proof in Beran (1994, Lemma 5.1). Since not all time series satisfying eq. (1) or (2) are invertible, the restriction to invertible processes is required.

THEOREM 1. *The time series Z_t satisfying (1) or (2) may be written in random shock form as $Z_t = A_t + \sum \psi_\ell A_{t-\ell}$, where $\psi_\ell \sim c_{\alpha,\psi}\, \ell^{-(1+\alpha)/2}$, $c_{\alpha,\psi} > 0$ and A_t is white noise. Assuming that Z_t is invertible, the inverted form may be written, $Z_t = A_t + \sum \pi_\ell Z_{t-\ell}$, where $\pi_\ell \sim c_{\alpha,\pi}\, \ell^{-(3-\alpha)/2}$, $c_{\alpha,\pi} > 0$ and A_t is white noise.*

PROOF. By the Wold Decomposition, any purely nondeterministic time series may be written in random shock form. Now assume the random shock coefficients specified in the theorem and we will derive (1). Assuming $\operatorname{var}(A_t) = 1$, $\gamma_Z(k) = \psi_k + \sum \psi_h \psi_{h+k}$

$$\begin{aligned}\gamma_Z(k) &\sim \psi_k + c_{\alpha,\psi}^2 \sum_{h=1}^{\infty} h^{-(1+\alpha)/2}(h+k)^{-(1+\alpha)/2} \\ &\sim \psi_k + c_{\alpha,\psi}^2 \int_1^\infty h^{-(1+\alpha)/2}(h+k)^{-(1+\alpha)/2} dh + R_k\end{aligned}$$

where the last step used the Euler summation formula (Graham, Knuth and Patashnik, 1989, 9.78, 9.80) and

$$R_k = \{-\frac{1}{2}F(h) + \frac{1}{12}F'(h) + \frac{\theta}{720}F'''(h)\}\Big|_1^\infty,$$

where $\theta \in (0,1)$ and $F(h) = h^{-(1+\alpha)/2}(h+k)^{-(1+\alpha)/2}$. It is easily shown that $k^\alpha R_k \to 0$ as $k \to \infty$. Hence,

$$\begin{aligned}\gamma_Z(k) &\sim \psi_k + c_{\alpha,\psi}^2 \int_1^\infty h^{-(1+\alpha)/2}(h+k)^{-(1+\alpha)/2} dh \\ &\sim \psi_k + k^{-\alpha} c_{\alpha,\psi}^2 \int_{1/k}^\infty x^{-\beta}(x+1)^{-\beta} dx,\end{aligned}$$

where $\beta = (1+\alpha)/2$. Using *Mathematica*,

$$\int_0^\infty x^{-\beta}(x+1)^{-\beta} dx = \frac{2^{2\beta}\,\Gamma(1-\beta)\,\Gamma(-\frac{1}{2}+\beta)}{4\sqrt{\pi}},$$

so (1) now follows with $c_{\alpha,\gamma} = c_{\alpha,\psi}^2 2^{\alpha-1}\,\Gamma((1-\alpha)/2)\,\Gamma(\alpha/2)/\sqrt{\pi}$, where $\Gamma(\bullet)$ is the gamma function. This shows that ψ_k is a possible factorization of γ_k and that sufficies to establish that $Z_t = A_t + \sum \psi_\ell A_{t-\ell}$.

For any stationary invertible linear process, Z_t,

$$\gamma_Z(k) = \sum_{h=1}^\infty \pi_h \gamma_Z(k-h). \tag{3}$$

Assume $\gamma_Z(k)$ satisfies eq. (1) and that $\pi_\ell \sim c_{\alpha,\pi}\ell^{-(3-\alpha)/2}$ then we will show that eq. (3) is satisfied.

$$\gamma_Z(k) \sim \gamma_Z(0)\pi_k + c\sum_{h=1}^{k-1} h^{-3/2+\alpha/2}(k-h)^{-\alpha} + c\sum_{h=k+1}^{\infty} h^{-3/2+\alpha/2}(h+k)^{-\alpha},$$

where $c = c_{\alpha,\pi}c_{\alpha,\gamma}$. Now $\gamma_Z(0)\pi_k/\gamma_Z(k) \sim 0$ so the first term will drop out. In the second term, for $k >> h$, $(k-h)^{-\alpha} \sim k^{-\alpha}$ and

$$\sum_{h=1}^{k-1} h^{-3/2+\alpha/2} \sim H_\beta \quad \text{as} \quad k \to \infty,$$

where $H_\beta = \sum_{h=1}^{\infty} h^{-\beta} < \infty$, $\beta = 3/2 - \alpha/2$. In the final term, when $h >> k$, $(h+k)^{-\alpha} \sim h^{-\alpha}$, so

$$\begin{aligned} \sum_{h=k+1}^{\infty} h^{-3/2+\alpha/2}(h+k)^{-\alpha} &\sim \sum_{h=k+1}^{\infty} h^{-3/2-\alpha/2} \\ &\sim \int_{k+1}^{\infty} h^{-3/2-\alpha/2} dh \\ &\sim (k+1)^{-(1+\alpha)/2}. \end{aligned}$$

Again the last step uses the Euler Summation Formula. Thus the final term is smaller asymptotically smaller than γ_k. This establishes the asymptotic equivalence of the left-hand side and the right-hand side of eq. (3) and the theorem since $\gamma_Z(k)$ uniquely determines the coefficients π_ℓ in the inverted model. $\triangle$

The FARMA model of order (p, q) (Granger and Joyeux, 1980; Hosking, 1981) may be defined by the equation,

$$\phi(B)(1-B)^d Z_t = \theta(B)A_t, \tag{4}$$

where $|d| < 0.5$, A_t is white noise with variance σ_A^2, $\phi(B) = 1 - \phi_1 B - \ldots - \phi_p B^p$, and $\theta(B) = 1 - \theta_1 B - \ldots - \theta_q B^q$. For stationarity and invertibility it is assumed that all roots of $\phi(B)\theta(B) = 0$ are outside the unit circle and $|d| < 0.5$. The series is strongly persistent or antipersistent according as $0 < d < 0.5$ or $-0.5 < d < 0$. The special case where $p = q = 0$ is known as fractionally differenced white noise.

Antipersistent series may arise in practice when modelling nonstationary time series. As suggested by Box & Jenkins (1976) a nonstationary time series can often be made stationary by differencing the series until stationarity is reached. Sometimes the resulting stationary time series may be usefully modelled by an antipersistent form of the FARMA model. An illustrative example is provided by the annual U.S. electricity consumption data for 1920–1970. Hipel and McLeod (1994, pp.154–159) modelled the square-root consumption using an ARIMA(0,2,1) but a better fit is obtained by modelling the second differences of the square-root consumption as fractionally differenced white noise with $d = -0.4477 \pm 0.1522$ sd. The AIC for the latter model is 1011.5 as compared with 1020.4. Diebold and Rudebusch (1989) and Beran (1995) also used this approach for modelling nonstationary data.

The determinant of the covariance matrix of n successive observations, $Z_t,\ t = 1,\dots,n$, is denoted by $G_Z(n) = \det(\gamma_Z(i-j))$. It will now be shown in Theorem 2 that for fractionally differenced white noise, $g_Z(n) = \sigma_A^{-2n} G_Z(n) \to \infty$ as $n \to \infty$, where $0 < \sigma_A^2 < \infty$, is the innovation variance given by Kolmogoroff's formula (Brockwell and Davis, eq. 5.8.1). In Theorems 7, 8 and 9 this result will be established for a more general family of processes. Since $g_Z(n)$ is the generalized variance of the process Z_t/σ_A, it will be referred to as the standardized generalized variance. Without loss of generality we will let $\sigma_A = 1$.

Theorem 2. *Let Z_t denote fractionally differenced white noise with parameter $d \in (-\frac{1}{2}, \frac{1}{2})$ and $d \neq 0$. Then $g_Z(n) \to \infty$.*

Proof. As in McLeod (1978), $g_Z(n) = \prod_{k=0}^{n-1} \sigma_k^2$, where σ_k^2 denotes the variance of the error in the linear predictor of Z_{k+1} using $Z_k, \dots, Z_1$. From the Durbin-Levinson recursion,

$$\sigma_k^2 = \begin{cases} \gamma_Z(0) & k = 0, \\ \sigma_{k-1}^2(1-\phi_{k,k}^2) & k > 0. \end{cases}$$

where $\phi_{k,k}$ denotes the partial autocorrelation function at lag k. For the special case $p = q = 0$ in (4), Hosking (1981) showed that $\phi_{k,k} = d/(k-d)$ and $\gamma_Z(0) = (-2d)!/(-d)!^2$. Using the Durbin-Levison recursion,

$$\sigma_k^2 = \frac{k!(k-2d)!}{(k-d)!^2}.$$

Applying the Stirling approximation to $\log(t!)$ for large t, $\log(t!) \sim (t+\frac{1}{2})\log(t) - t + \frac{1}{2}\log(2\pi)$, yields $\log \sigma_k^2 \sim a(k)$, where

$$a(k) = (k + \frac{1}{2}) \log \frac{k(k-2d)}{(k-d)^2} + 2d \log \frac{k-d}{k-2d}.$$

Since σ_k^2, is a monotone decreasing sequence and for $d \neq 0$, $\sigma_k^2 > 1$, it follows that $\log(\sigma_k^2)$ is a positive monotone decreasing sequence. By Stirling's approximation $\log(\sigma_k^2)/a(k) \to 1$ as $k \to \infty$. So for large k, $a(k)$ must be a monotone decreasing sequence of positive terms. Expanding $a(k)$ and simplifying

$$\begin{aligned} a(k) &= (k+\frac{1}{2})\log(1-\frac{2d}{k}) + 2(k+\frac{1}{2})\log\{1+\frac{d}{k}+\left(\frac{d}{k}\right)^2+\ldots\} + 2d\log(1+\frac{d}{k-2d}) \\ &= \frac{d^2}{k} + O(\frac{1}{k^2}), \end{aligned}$$

where the expansion $\log(1+x) = x + x^2/2 + x^3/3 + \ldots$, $|x| < 1$ as been used. Hence,

$$ka(k) \to d^2 \text{as } k \to \infty \tag{5}$$

and by the Theorem given by Knopp (1951, §80, p.124), $\sum a(k)$ diverges for $d \neq 0$. So for $d \neq 0$, $\sum \log(\sigma_k^2)$ diverges and consequently so does $g_Z(n)$. △

Eq. (5) shows that $\sigma_k^2 = 1 + O(k^{-1})$ which implies σ_k^2 decays very slowly. The divergence of $g_Z(n)$ can be slow. See Table I.

TABLE I.

GENERALIZED VARIANCE, $g_Z(n)$, FOR $n = 10^k$, $k = 0, 1, \ldots, 7$

OF FRACTIONALLY DIFFERENCED WHITE NOISE, Z_t, WITH PARAMETER d.

d	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
−0.4	1.1831	1.6225	2.3318	3.3685	4.8688	7.0375	10.1725	14.7059
−0.1	1.0145	1.0366	1.0607	1.0854	1.1107	1.1365	1.1630	1.1901
0.1	1.0195	1.0434	1.0678	1.0927	1.1181	1.1442	1.1708	1.1990
0.4	2.0701	3.1588	4.5923	6.6417	9.6009	13.8775	20.0591	28.9951

2. HYPERBOLIC DECAY TIME SERIES

The stationary, purely nondeterministic time series, Z_t, is said to be a hyperbolic decay time series with decay parameter α, $\alpha \in (0,2)$, $\alpha \neq 1$, if for large k

$$\gamma_Z(k) \sim c_{\alpha,\gamma}\, k^{-\alpha}, \tag{6}$$

where $c_{\alpha,\gamma} > 0$ for $\alpha \in (0,1)$ and $c_{\alpha,\gamma} < 0$ for $\alpha \in (1,2)$. When $\alpha \in (1,2)$ the time series is said to be antipersistent. As shown in the next theorem, antipersistent time series have a spectral density function which decays rapidly to zero near the origin. The term antipersistent was coined by Mandelbrot (1983) for FGN processes with Hurst parameter, $0 < H < 1/2$. Hyperbolic decay time series include both FGN time series with parameter $H = 1 - \alpha/2$, $H \in (0,1)$, $H \neq 1/2$ and FARMA time series with parameter $d = 1/2 - \alpha/2$, $d \in (-1/2, 1/2)$, $d \neq 0$.

THEOREM 3. *The spectral density function of hyperbolic decay time series satisfies (2).*

PROOF. Beran (1994) established this result when $\alpha \in (0,1)$ as was noted above in eq. (2). However the Theorem of Zygmund (1968, §V.2) used by Beran (1994, Theorem 2.1) does not apply to the case where $\alpha \in (1,2)$.

Let Y_t have the spectral density, $f_Y(\lambda) = c_{\alpha,f}\lambda^{\alpha-1}$, $\alpha \in (1,2)$.

$$\begin{aligned}\gamma_Y(k) &= 2\int_0^{\pi} c_{\alpha,f}\lambda^{\alpha-1}\cos(\lambda k)d\lambda \\ &= 2c_{\alpha,f}k^{-\alpha}\int_0^{k\pi} u^{\alpha-1}\cos(u)du,\end{aligned}$$

Using *Mathematica*,

$$\int_0^{\infty} u^{\alpha-1}\cos(u)du = \frac{\sqrt{\pi}\,\Gamma(\frac{\alpha}{2})}{(\frac{1}{4})^{\frac{\alpha-1}{2}}\,\Gamma(\frac{1-\alpha}{2})}$$

and so $\gamma_Y(k) \sim c_{\alpha,\gamma}k^{-\alpha}$, where $c_{\alpha,\gamma} = 2c_{\alpha,f}\sqrt{\pi}\,\Gamma(\frac{\alpha}{2})/\{(\frac{1}{4})^{\frac{\alpha-1}{2}}\,\Gamma(\frac{1-\alpha}{2})\} < 0$.

Assume $f_Z(\lambda)$ satisfies eq. (2) and we will derive (6). Since $f_Z(\lambda)/(c_f\lambda^{\alpha-1}) \to 1$ as $\lambda \to 0$, there exists λ_0 such that for all $\lambda < \lambda_0$, $c_f\lambda^{\alpha-1} < 1$ and $|f_Z(\lambda)/(c_f\lambda^{\alpha-1}) - 1| <$

$\epsilon/(2\pi)$. Hence for all $\lambda < \lambda_0$, $|f_Z(\lambda) - f_Y(\lambda)| < \epsilon/(2\pi)$. Consider the systematically sampled series, $Z_{t,\ell} = Z_{t\ell}$ for $\ell \geq 1$. Then $Z_{t,\ell}$ has spectral density function, $f_Z(\lambda/\ell)$. Let $L = \pi/\lambda_0$. Then $|f_Z(\lambda/\ell) - f_Y(\lambda)| < \epsilon/(2\pi)$ for $\lambda \in (0, \pi)$ provided that $\ell > L$. Hence for any $\ell > L$,

$$\begin{aligned}|\gamma_Z(kl) - \gamma_Y(k)| &< 2\int_0^{\pi} |\cos(\lambda k)||f_Z(\lambda/\ell) - f_Y(\lambda)|d\lambda \\ &< 2\int_0^{\pi} |f_Z(\lambda/\ell) - f_Y(\lambda)|d\lambda \\ &< \epsilon.\end{aligned}$$

This shows (2) implies (6). Since the spectral density uniquely defines the autocovariance function, the theorem follows. $\triangle$

Hyperbolic decay time series are self-similar: aggregated series are hyperbolic with the same parameters as the original.

THEOREM 6. *Let Z_t satisfy eq. (6) then so does Y_t, where $Y_t = \sum_{j=1}^{m} Z_{(t-1)m+j}/m$ and m is any value.*

PROOF. For large ℓ,

$$\begin{aligned}\gamma_Y(\ell) &= m^{-2}\mathrm{cov}(\sum_{h=1}^{m} Z_{(t-1)m+h}, \sum_{k=1}^{m} Z_{(t-1)m+k+\ell}) \\ &\sim m^{-2}\sum_{h=1}^{m}\sum_{k=1}^{m} c_{\alpha,\gamma}(k+\ell-h)^{-\alpha} \\ &\sim m^{-2}\sum_{h=1}^{m}\sum_{k=1}^{m} c_{\alpha,\gamma}\ell^{-\alpha}(1+\frac{(k-h)}{\ell})^{-\alpha} \\ &\sim c_{\alpha,\gamma}\ell^{-\alpha}.\end{aligned}$$

3. DUALITY

Duality has provided insights into linear time series models (Finch, 1960; Pierce, 1970; Cleveland, 1972; Box and Jenkins, 1976; Shaman, 1976; McLeod, 1977, 1984). In general, the dual of the stationary invertible linear process $Z_t = \psi(B)A_t$ is defined to be $\psi(B)\ddot{Z}_t = A_t$, where $\psi(B) = 1 + \psi_1 B + \psi_2 B^2 + \ldots$ and B is the backshift operator on t. Equivalently, if Z_t has spectral density $f_Z(\lambda)$ then the dual has spectral density proportional to $1/f_Z(\lambda)$ with the constant of proportionality determined by the innovation variance. Thus in the case of a FARMA(p, q) with parameter d the dual is a FARMA(q, p) with parameter $-d$. The next theorem generalizes this to the hyperbolic case.

THEOREM 4. *The dual of a hyperbolic decay time series with decay parameter α is another hyperbolic decay series with parameter decay parameter $2 - \alpha$.*

PROOF. The spectral density near zero of the dual of a hyperbolic decay time series with parameter α is $1/(c_f \lambda^{\alpha-1}) = c_f^{-1} \lambda^{(2-\alpha)-1}$ which implies a hyperbolic process with parameter $2 - \alpha$. △

THEOREM 5. *The time series Z_t satisfying (6) may be written in random shock form as $Z_t = A_t + \sum \psi_\ell A_{t-\ell}$ where $\psi_\ell \sim c_{\alpha,\psi} \ell^{-(1+\alpha)/2}$ and $c_{\alpha,\psi} > 0$ for $\alpha \in (0, 1)$ and $c_{\alpha,\psi} < 0$ for $\alpha \in (1, 2)$ and in inverted form as $Z_t = A_t + \sum \pi_\ell Z_{t-\ell}$ where $\pi_\ell \sim c_{\alpha,\pi} \ell^{-(3-\alpha)/2}$ and $c_{\alpha,\pi} > 0$ for $\alpha \in (0, 1)$ and $c_{\alpha,\pi} < 0$ for $\alpha \in (1, 2)$*

PROOF. The case $\alpha \in (0, 1)$ was established in Theorem 1. When $\alpha \in (1, 2)$ the random shock coefficients are given by

$$\begin{aligned} \psi_\ell &\sim -c_{2-\alpha,\pi} \ell^{-\{3-(2-\alpha)\}/2} \\ &\sim c_{\alpha,\psi} \ell^{-(1+\alpha)/2}, \end{aligned}$$

where $c_{\alpha,\psi} = -c_{2-\alpha,\pi}$. Similarly for the inverted form. △

4. GENERALIZED VARIANCE

For ARMA processes, Z_t, $\lim g_Z(n)$ is finite and has been evaluated by Finch (1960) and McLeod (1977). McLeod (1977, eq. 2) showed $g_Z(n) = m_Z + O(r^n)$, where $r \in (0, 1)$. The evaluation of this limit uses the Theorem of Grenander and Szegö (1984, §5.5) which only applies to the case where the spectral density, $f_Z(\lambda), \lambda \in [0, 2\pi)$ satisfies the

Lipschitz condition $|f'_Z(\lambda_1) - f'_Z(\lambda_2)| < K|\lambda_1 - \lambda_2|^\zeta$, for some $K > 0$ and $0 < \zeta < 1$. Since when $\alpha \in (0,1)$, $f'_Z(\lambda)$ is unbounded, this condition is not satisfied.

LEMMA 1. *Let X_t and Y_t be any independent stationary processes with positive innovation variance and let $Z_t = X_t + Y_t$. Then $G_Z(n) > G_X(n)$*

PROOF. This follows directly from the fact that the one-step predictor error variance of Z_t can not be less than that of X_t. $\triangle$

THEOREM 7. *Let Z_t denote a strongly persistent time process defined in eq. (2). Then $g_Z(n) \to \infty$.*

PROOF. Since $Z_t = \sum \psi_k A_{t-k}$, where A_t is white noise with unit variance, we can find a q such that the process Y_t, where

$$Y_t = \sum_{k=q+1}^{\infty} \psi_k A_{t-k},$$

has all autocovariances nonnegative and satisfying eq. (1). By using the comparison test for a harmonic series, it must be possible to find an N such that for $n > N$, the covariance matrix $\Gamma_Y(n)$ has every row-sum greater than Ξ, for any $\Xi > 0$. It then follows from Frobenius Theorem (Minc and Marcus, 1964, p.152) that the largest eigenvalue of $\Gamma_Y(n)$ tends to ∞ as $n \to \infty$. Assume now that $\inf f_Y(\lambda) = m$ where $m > 0$ and let m_n denote the smallest eigenvalue of $\Gamma_Y(n)$ and let ζ_n denote the corresponding eigenvector. Then

$$\begin{aligned} m_n &= m_n \zeta_n' \zeta_n \\ &= \zeta_n' \Gamma_Y(n) \zeta_n \\ &= \int_{-\pi}^{\pi} \sum_h \sum_\ell \zeta_{n,h} \zeta_{n,\ell} e^{-i\lambda(h-\ell)} f(\lambda) d\lambda \\ &\geq 2\pi m. \end{aligned}$$

So $m_n \geq 2\pi m$ and hence $g_Y(n) \to \infty$ as $n \to \infty$. By Lemma 1, $g_Z(n) \to \infty$ also.

For the more general case where $m = 0$, consider a process with spectral density function $f(\lambda) + \epsilon$, where $\epsilon > 0$. Let $g_\epsilon(n)$ denote the standardized covariance determinant of n successive observations of this process. So $g_\epsilon(n) \to \infty$ as $n \to \infty$ for every $\epsilon > 0$. The autocovariance function corresponding to $f(\lambda) + \epsilon$ is

$$\gamma_\epsilon(k) = \begin{cases} \gamma_Z(0) + 2\pi\epsilon & k = 0, \\ \gamma_Z(k) & k \neq 0. \end{cases}$$

By continuity of the autocovariance function with respect to ϵ, $\lim g_\epsilon(n) \to g_Z(n)$ as $\epsilon \to 0$. Let $\Xi > 0$ be chosen as large as we please and let $\delta > 0$. Then for any $\epsilon > 0$ there exists an $N(\epsilon)$ such that for all $n \geq N(\epsilon)$, $g_\epsilon(n) > \Xi + \delta$. By continuity, there exists an ϵ_0 such that $g_Z(N(\epsilon_0)) > g_{\epsilon_0}(N(\epsilon_0)) - \delta$. Hence $g_Z(N(\epsilon_0)) > \Xi$. Since $g_Z(n + 1) = g_Z(n)\sigma_n^2$,where $\sigma_n^2 > 1$ is the variance of the error of the linear predictor of Z_{n+1} given $Z_n, \ldots, Z_1$ we see that $g_Z(n)$ is nondecreasing. It follows that $g_Z(n) > \Xi$ for all $n > N(\epsilon_0)$. $\triangle$

Using a Theorem of Grenander and Szegö (1984) this result is easily generalized to any stationary time series, Z_t, for which $\sum \gamma_Z(k) = \infty$.

THEOREM 8. *Let Z_t denote a time series for which $f_Z(\lambda) \to \infty$ as $\lambda \to 0$. Then $g_Z(n) \to \infty$.*

PROOF. From eq. (10) of Grenander and Szegö (1984, §5.2), as $n \to \infty$, the largest eigenvalue of $\sigma_a^{-2}\Gamma_Z(n)$ approaches $\sup f_Z(\lambda) = \infty$ while the smallest eigenvalue approaches $2\pi m$, where $m = \inf\ f(\lambda)$. Note that Grenander and Szegö's eq. (10) of §5.2, applies directly to unbounded spectral densities as is pointed by Grenander and Szegö in the sentence immediately following eq. (10), §5.2. If it is assumed that $m > 0$, then the largest eigenvalue tends to infinity and the smallest one is bounded by $2\pi m$ as $n \to \infty$. Hence, $g_Z(n) \to \infty$ for this special case. The more general case where $m = 0$ is handled as in Theorem 7. $\triangle$

In the case of ARMA models, the asymptotic covariance determinant of the dual and primal are equal (Finch, 1960). Since the hyperbolic decay time series are approximated by high order AR and MA models, it might be expected that this property holds for hyperbolic series too. Theorem 9 which uses Lemma 2 proves that this is the case.

LEMMA 2. *Let* $X_t = A_t + \sum_1^\infty \psi_\ell A_{t-\ell}$. *Let* $X_t(q) = A_t + \sum_1^q \psi_\ell A_{t-\ell}$, *and let* $g_q(n)$ *denote its standardized covariance determinant. Then for any* $\ell > 0$, $g_{q+\ell}(n) \geq g_q(n)$.

PROOF. This follows directly from the fact that the one-step predictor error variance of $X_t(q+\ell)$ can not be less than that of $X_t(q)$. $\triangle$

THEOREM 9. *For hyperbolic decay antipersistent time series,* Z_t, $g_Z(n) \to \infty$.

PROOF. Since the dual of the antipersistent time series Z_t with parameter $2-\alpha$, $\alpha \in (0,1)$ is a strongly persistent time series $\ddot{Z}_t$ with parameter α, $\ddot{Z}_t$ may be represented in inverted form, $\ddot{Z}_t = A_t + \sum \pi_k \ddot{Z}_{t-k}$, where A_t is white noise and for large k, $\pi_k \sim c_\pi k^{-(3-\alpha)/2}$. So the antipersistent time series Z_t can be written, $Z_t = A_t - \sum \pi_k A_{t-k}$. Let $\ddot{g}_L(n)$ and $g_L(n)$ denote the covariance determinant of n successive observations in the AR(L) and MA(L) approximation to $\ddot{Z}_t$ and Z_t

$$\ddot{Z}_t(L) = A_t + \sum_{k=1}^{L} \pi_k \ddot{Z}_{t-k}(L)$$

and

$$Z_t(L) = A_t - \sum_{k=1}^{L} \pi_k A_{t-k}.$$

By Theorem 7, for any $\Xi > 0$ and $\delta > 0$ there exists an N_1 such that for $n > N_1$, $g_{\ddot{Z}}(n) > \Xi + \delta$. Since $\ddot{g}_k(n) \to \ddot{g}_Z(n)$ as $k \to \infty$ there exists a $K_1(n)$ such that $\ddot{g}_k(n) > \ddot{g}_Z(n) - \delta > \Xi$ for $k > K_1(n)$. From McLeod (1977), $\ddot{g}_k(n) = \ddot{g}_k(k)$ for $n \geq k$. Hence for any $n > N_1$, $\ddot{g}_k(m) > \ddot{g}_Z(n) - \delta > \Xi$ for $k > K_1(n)$ and $m \geq k$. So $\ddot{g}_k(m) \to \infty$ as $k \to \infty$ and $m \geq k$.

Hence there exists K_2 such that $\ddot{g}_k(n) > \Xi + \delta$ for $k > K_2$ and $n \geq k$. For any k, $g_k(n) = \ddot{g}_k(n) + O(r^n)$, where $0 < r < 1$ (McLeod, 1977). Let $k > K_2$. Then there exists an $N_2(k)$ such that for all $n > N_2(k)$, $g_k(n) > \ddot{g}_k(n) - \delta > \Xi$. So $g_k(n) \to \infty$ as $k \to \infty$ and $n \geq k$.

For any n, $g_k(n) \to g_Z(n)$ as $k \to \infty$. So for any n there exists a $K_3(n)$ such that $g_Z(n) > g_k(n) - \delta$ for all $k > K_3(n)$. We have already established that there exists a

K_4 such that $g_k(n) > \Xi + \delta$ for $k > K_4$ and $n \geq k$. Holding n fixed for the moment, let $h > k$. By Lemma 2, $g_h(n) \geq g_k(n)$. By continuity since $h > K_4$, $g_Z(n) > g_h(n) - \delta$. Since $g_h(n) > \Xi + \delta$ it follows that $g_Z(n) > \Xi$. This establishes that $g_Z(n) \to \infty$ as $n \to \infty$. $\triangle$

5. CONCLUDING REMARKS

Theorems 7 and 9 show that hyperbolic decay time series, even antipersistent ones, exhibit a type of long-range dependence. The asymptotic standardized generalized variance is infinite. This implies that the variance of the one-step linear predictor based on the last k observations decays very slowly as compared with the ARMA case where the decay to the innovation variance occurs exponentially fast. Theorem 8 shows that this is a more general notion of long-range dependence than the customary one.

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