Parsimony, Model Adequacy And Periodic Correlation In Time Series Forecasting

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Summary

The merits of the modelling philosophy of Box & Jenkins (1970) are illustrated with a summary of our recent work on seasonal river flow forecasting. Specifically, this work demonstrates that the principle of parsimony, which has been questioned by several authors recently, is helpful in selecting the best model for forecasting seasonal river flow. Our work also demonstrates the importance of model adequacy. An adequate model for seasonal river flow must incorporate seasonal periodic correlation. The usual autoregressive-moving average (ARMA) and seasonal ARMA models are not adequate in this respect for seasonal river flow time series. A new diagnostic check, for detecting periodic correlation in fitted ARMA models is developed in this paper. This diagnostic check is recommended for routine use when fitting seasonal ARMA models. It is shown that this diagnostic check indicates that many seasonal economic time series also exhibit periodic correlation. Since the standard forecasting methods are inadequate on this account, it can be concluded that in many cases, the forecasts produced are sub-optimal. Finally, a limitation of the arbitrary combination of forecasts is also illustrated. Combining forecasts from an adequate parsimonious model with an inadequate model did not improve the forecasts whereas combining the two forecasts of two inadequate models did yield an improvement in forecasting performance. These findings also support the model building philosophy of Box & Jenkins. The non-intuitive findings of Newbold & Granger (1974) and Winkler & Makridakis (1983) that the apparent arbitrary combination of forecasts from similar models will lead to forecasting performance is not supported by our case study with river flow forecasting.

Keywords: Combined Forecasts; Diagnostic Check for Periodic Correlation; Forecasting Seasonal Time Series; Model Adequacy; Parameter Parsimony.

1 Introduction

The main purpose of this paper is to discuss some general statistical principles which are elucidated by our recent work in river flow forecasting (McLeod et al., 1987; Noakes et al., 1985; Thompstone et al., 1985a). Also based on these case studies, a new diagnostic check for periodic correlation in the residuals of fitted ARMA models is developed. This diagnostic check is suitable for routine use when fitting seasonal ARMA models.

Briefly our experience with river flow time series suggests that the best forecasting results are obtained by following the general model building philosophy implicit in Box & Jenkins (1970) with suitable modifications and improvements. In general terms, this approach is iterative and advocates choosing the most parsimonious adequate statistical model. Two basic principles of special relevance are:

Principle 1: Model Adequacy.

The model is considered adequate if it incorporates all relevant information and if when calibrated to the data, no important significant departures from the statistical assumptions made can be found.

Principle 2: Model Parsimony.

The principle of parsimony means that the simplest possible model should be chosen.

One can view the problem of statistical modelling as choosing an adequate statistical model which is the most parsimonious. In mathematical programming terminology we could say that the problem of statistical modelling has an objective function which is to minimize the model complexity (Model Parsimony) subject to the constraint of Model Adequacy.

In §2 the results of a case study of forecasting monthly river flow time series is summarized. Here the importance of incorporating periodic correlation in the forecasting model is demonstrated. For a seasonal time series denoted by $z_{r,p}$ where r denotes the year and p denotes the seasonal period, the periodic correlation coefficient is defined by

$$\rho_m(\ell) = \frac{\gamma_m(\ell)}{\sqrt{(\gamma_m(0)\gamma_{m-\ell}(0))}},$$

where

$$\gamma_m(\ell) = \mathrm{Cov}(z_{r,m}, z_{r,m-\ell}).$$

The concept of periodically correlated processes was introduced by Gladyshev (1961). The first application of periodic time series models seems to have been by hydrologists Thomas & Fiering (1962). Since that time there have been very extensive developments in the theory and applications of periodically correlated time series. For a review of the probabilistic literature on periodically correlated processes, see Yaglom (1986, §26.5; 1987). Miamee (1990) and Sakai (1991) have derived new theoretical results and conditions on the spectral density function of periodically correlated time series. On the statistical methodology side, contributions to periodically correlated time series modelling have been made by Jones & Brelsford (1967), Moss & Bryson (1974), Pagano (1978), Cleveland & Tiao (1979), Troutman (1979), Tiao & Grupe (1980), Sakai (1982), Dunsmuir, W. (1983), Thompstone et al. (1985b), Vecchia (1985a, 1985b), Vecchia et al. (1985), Li & Hui (1988), Jiménez et al. (1989), Hurd & Gerr (1991), Osborn, D.R. (1991) and Vecchia, A.V. & Ballerini, R. (1991). Periodic time series models are often used for modelling seasonal time series – especially environmetric series. However, several other interesting applications include multiple spectral estimation (Newton, 1982) and multichannel signal processing (Sakai, 1990).

In some situations, as in the case study in §3, a comprehensive modelling approach which satisfies both adequacy and parsimony principles may not be practical either for reasons of expediency or because a suitable model cannot be found with available methodology. In this case, we have found combined forecasts to be useful. On the other hand, if a good model can be found, our experience suggests that the forecast cannot be significantly improved by combining it with forecasts from models which are less parsimonious or less adequate. This latter result is at variance with the results reported by Winkler & Makridakis (1983) and Newbold & Granger (1974). Perhaps this is due to the fact that the river flow time series used in our studies are generally longer and more homogeneous than the economic series used by the aforementioned authors. The skill of the modeller in developing an adequate model could also be a factor.

In order to make the hydrological data sets used in the case studies referred to in this paper readily accessible to other researchers, all data is available in the statlib computer archive. One may obtain an electronic copy of this data by sending an e-mail message to statlib@lib.stat.cmu.edu. The message should be: `send riverflows from datasets`.

2 Monthly River Flow Case Study

The data in this case study (Noakes et al., 1985) consisted of thirty mean monthly river flows for periods of from 37 to 64 years. Various models and model selection procedures were used to calibrate a model to each data set omitting the last three years of data. The one-step ahead forecasts were then compared for the last three years (36 values). The best forecasts as judged by the root mean-square error and other criteria were obtained with the family of periodic autoregressive models.

The periodic autoregression model equation may be written

$$\phi_m(B)(Z_{r,m} - \mu_m) = a_{r,m} \tag{1}$$

where $Z_{r,m}$ denotes the logarithmic flow for the r^{th} year and m^{th} month, μ_m denotes the corresponding monthly mean, $a_{r,m}, r = 1, 2, \ldots, m = 1, 2, \ldots, 12$ are a sequence independent normal random numbers with mean zero and variance, σ_m^2, and

$$\phi_m(B) = 1 - \phi_{m,1}B - \ldots - \phi_{m,p_m}B^{p_m} \tag{2}$$

where B is the backshift operator on t, where $t = 12(r-1) + m$. Several model selection techniques were used to select p_m $(m = 1, 2, \ldots, 12)$. It was found that a periodic autoregression which was determined by choosing p_m as small as possible to achieve an adequate fit gave the best forecasts. This was accomplished by initially determining p_m based on a plot of the periodic partial autocorrelation function and then checking the adequacy of the fitted model. Our approach is thus a natural extension of that of Box & Jenkins (1970).

On the other hand, a subset periodic autoregression approach was found to produce comparatively very poor forecasts. In this approach, for each period all possible autoregressions with some parameters constrained to zero and with $p_m = 12$ were examined (2^{12} possibilities) and the best model was selected with the Akaike Information Criterion (Akaike, 1974) as well as the Bayes Information Criterion (Schwarz, 1978). It was also noticed that the resulting models were always less parsimonious than that selected by the first approach.

The seasonal ARMA model developed by Box & Jenkins (1970, Ch. 9) did not perform very well either. In this case, the diagnostic check, developed in the next §4, indicates that this is due to model inadequacy.

The periodic autoregression and seasonal ARMA represent quite different families of time series models. Not only do the models differ in the correlation structure but the specification of seasonality is purely stochastic in the seasonal ARMA model and purely deterministic in the case of the periodic autoregression. Moreover neither specification is likely to be absolutely correct. Thus although the periodic autoregression model forecasted best and was considered to represent a more valid statistical model, it might be thought from the experience reported by Newbold & Granger (1974) and Winkler & Makridakis (1983) that combining the periodic autoregression and seasonal ARMA forecasts would be helpful. As shown in McLeod et al. (1987) this is not the case. In particular with method 1 of Winkler & Makridakis (1983, p. 152) the periodic autoregression forecast had a smaller mean square error at least 17 times out of 30. Thus combined forecasts cannot be recommended in this situation.

3 Quarter-Monthly River Flow Case Study

The object of this study (Thompstone et al., 1985a) was to obtain one-step-ahead forecasts of the quarter-monthly, i.e. almost weekly, inflows to the Lac St. Jean reservoir system operated by Alcan Limited. Complete time series on past quarter-month inflows, precipitation and snowmelt in the river basin were available for 30 years. A Box-Jenkins multiple transfer-function noise model with precipitation and snowmelt as inputs was found to provide an adequate fit to the deseasonalized data in many respects except that it did not account for the periodic correlation effect. A periodic autoregression model was also fit but this model did not take into account the covariates precipitation and snowmelt. It could be suggested that at this stage a periodic-transfer-function noise model should be developed to take into account both factors. However such a model could easily involve too many parameters and, in any case, it was not possible to calibrate it with our existing computer software. Perhaps future work will result in a suitable model. Finally, a third model which was a semi-theoretical hydrological model which incorporates various hydrological and meteorological information in a conceptual model of river flow. The conceptual modelling approach has been strongly advocated by certain hydrologists who feel that time series methods are too empirical.

All three models were calibrated on data for 27 years and then used to produce one-step-ahead forecasts over the next three years (144 periods). The root mean square error for transfer-function noise, periodic autoregression and conceptual model for forecasting logarithmic flows were respectively 0.2790, 0.3009 and 0.3894. When the forecasts were combined by simple averaging the root mean square error dropped to 0.1355. More sophisticated combination techniques were found to lead to even further improvements.

It is interesting to note that the empirical time series approach outperformed the more theoretical conceptual approach which has been strongly advocated by some hydrologists. A similar phenomenon with macro-economic time series forecasting as previously been found (Naylor et al., 1972).

4 A New Diagnostic Check For Periodic Autocorrelation

The seasonal ARMA model of order $(p, d, q)(p_s, d_s, q_s)_s$ may be written

$$\Phi(B^s)\phi(B)\nabla_s^{d_s}\nabla^d Z_t = \Theta(B^s)\theta(B)a_t, \tag{3}$$

where Z_t is the observation at time t and a_t is a sequence of independent normal random variables with mean zero and variance σ^2. For monthly time series $s = 12$ and $t = 12(r-1) + m$, where r and m represent the year and month respectively. The polynomials $\Phi(B^s)$, $\phi(B)$, $\Theta(B^s)$ and $\theta(B)$ of degrees p_s, p, q_s and q specify the autoregressive and moving average components of the model. The terms $\nabla_s = 1 - B^s$ and $\nabla = 1 - B$ represent the seasonal and non-seasonal differencing operators. Using standard model selection techniques (Box & Jenkins, 1970; Hipel et al., 1977) it was found that most monthly river flow time series could be tentatively modelled as a seasonal ARMA model of order $(p, 0, 1)(0, 1, 1)_{12}$, where $p = 0$, 1 or 2. The diagnostic check described below can be used to check for model inadequacy due to periodic correlation in the residuals of such fitted models.

The residual periodic autocorrelation at lag $k \geq 1$ may be written

$$\hat{r}_m(k) = \frac{\sum_r \hat{a}_{r,m}\hat{a}_{r,m-k}}{\sqrt{\sum_r \hat{a}_{r,m}^2 \sum_r \hat{a}_{r,m-k}^2}}, \tag{4}$$

where $\hat{a}_{r,m}$ denotes the seasonal ARMA model residual for period $t = 12(r-1) + m$ $(r = 1, \ldots, N; m = 1, \ldots, 12)$. If the seasonal ARMA model is adequate then using the methodology in McLeod (1978) it can be shown for any fixed $M \geq 1$, $\sqrt{N}\hat{\mathbf{r}}^{(m)} = \sqrt{N}(\hat{r}_m(1), \ldots, \hat{r}_m(M))$ is asymptotically normal with mean zero and covariance matrix $(1_M - Q/12)/N$, where 1_M is the identity matrix of order M and $Q = XI^{-1}X^T$, where X and I are given in eq. (44) of McLeod (1978). Moreover, $\sqrt{N}\mathbf{r}^{(m)}$ and $\sqrt{N}\mathbf{r}^{(m')}$ are asymptotically independent when $m \neq m'$. Since the diagonal elements of Q are all less than one, it follows that to a good approximation, $\hat{r}_m(1), m = 1, \ldots, 12$ are jointly normally distributed with mean vector zero, diagonal covariance matrix and $\mathrm{Var}(\hat{r}_m(1)) = N^{-1}$. A diagnostic check for detecting periodic autocorrelation in seasonal ARMA model residuals is given by

$$S = N \sum_{m=1}^{12} \hat{r}_m^2(1) \tag{5}$$

which should be approximately χ^2-distributed on 12 df.

As a check on the asymptotic approximation involved, a brief simulation experiment was performed. A $(1,0,0)(0,0,0)_{12}$ model with $\phi_1 = -0.9, -0.6, -0.3, 0.3, 0.6$ and 0.9 was simulated. Table 1 summarizes the results on S for one thousand simulations with $N = 17$. The empirical significance level of a nominal 5% test was estimated by counting the number of times that S exceeded 21.0261. From Table 1, the approximation is seen to be adequate for practical purposes. In further experiments with $N = 34$ and 68, the approximation was seen to improve although the empirical significance level was still slightly less than 0.05 in all cases. This suggests that in general the significance will be slightly overestimated. For example, if the observed value of S indicates significant periodic correlation at the 5% level, the true significance level will be slightly less than 5%.

[Table 1 here]

The data on the Saugeen River (1915–1976) is illustrative of the usefulness of this new diagnostic check. A $(1,0,1)$ $(0,1,1)_{12}$ model was fit to the logarithmic flows and passed all diagnostic checks given in Box & Jenkins (1970). However, it was found that $S = 59.6$ indicating very significant residual periodic correlation. As indicated in the next section, it appears that many seasonal economic time series also exhibit such periodic residual correlations.

Table 1

Empirical mean, variance and significance level of S with $N = 17$ in 1000 simulations using a nominal 5% test.

ϕ_1	Mean	Variance	Significance level
–0.9	11.9	19.3	0.032
–0.6	11.5	18.6	0.025
–0.3	11.3	19.0	0.023
0.0	10.9	16.6	0.016
0.3	11.1	19.7	0.027
0.6	11.5	18.6	0.026
0.9	11.7	19.7	0.030

5 Application to Forecasting Economic Time Series

Many seasonal economic time series may exhibit periodic correlation which most of the standard approaches do not take into account. The diagnostic check of §4 may be applied routinely when fitting seasonal ARMA models. Table 2 shows the results of testing the seasonal ARMA models fitted by Miller & Wichern (1977, p.432) to four Wisconsin series. It is seen that in two out of the four series there is very significant periodic correlation. In these cases, models which take this correlation into account may be expected to produce improved forecasts. A comprehensive new approach to the modelling and forecasting of such series is given by McLeod (1992).

[Table 2 here]

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Table 2

Diagnostic Test For Residual Periodic Correlation

For Four Wisconsin Series From Miller & Wichern

Category	S	*d.f.*	Significance level
Food Products	25.36	12	0.013
Fabricated Metals	36.8	12	0.0002
Transportation Equipment	11.6	12	0.478
Trade	6.98	12	0.859

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Résumé

Les mérites de la philosophie de la modélisation de Box et Jenkins (1970) sont illustrées par un résumé de nos recherches récentes sur la prévision de saisonniers débits en rivière. En particulier, nos résultats démontrent que le principe de la parcimonie, que plusieurs auteurs ont mis en question, est utile à la sélection du meilleur modèle pour prévoir les saisonniers débits en rivière. Nos recherches démontrent l'importance de la compétence d'un modèle. Un modèle adéquat de saisonniers débits en rivière doit incorporer la corrélation périodique et saisonnière. Les modèles autorégressifs à moyenne mobile (ARMA) habituels et les modèles ARMA saisonniers ne sont pas adéquats à cet égard pour les séries de saisonniers débits en rivière. Dans cet article, on développe une nouvelle méthode pour déceler la corrélation périodique dans les modèles ARMA ajustés. Cette méthode est à utiliser habituellement dans l'ajustement des modèles ARMA saisonniers. Cette méthode indique que beaucoup de séries chronologiques économiques font preuve de la corrélation périodique aussi. Puisque les méthodes ordinaires de prévision ne sont pas adéquats, on peut conclure que dans beaucoup de cas les prévisions produites sont moins qu'optimales. En dernier lieu, une limitation à la combinaison arbitraire des prévisions est illustrée aussi. La combinaison des prévisions d'un modèle parcimonieux et adéquat avec celles d'un modèle inadéquat n'améliora pas les prévisions. Cependant, le fait de combiner les deux prévisions de deux modèles inadéquats produisit une amélioration de la performance de la prévision. Ces résultats appuient aussi la philosophie de la modélisation de Box et Jenkins. Les résultats non-intuitifs de Newbold et Granger (1974) et de Winkler et Makridakis (1983) indiquent que la combinaison apparente et arbitraire de prévisions de modèles semblables menèra à la performance des prévisions. Cette conclusion n'est pas soutenue par notre étude de cas portant sur la prévision des débits en rivière.