# Diagnostic Checking Periodic Autoregression Models With Application


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**Abstract.** An overview of model building with periodic autoregression (PAR) models is given emphasizing the three stages of model development: identification, estimation and diagnostic checking. New results on the distribution of residual autocorrelations and suitable diagnostic checks are derived. The validity of these checks is demonstrated by simulation. The methodology discussed is illustrated with an application. It is pointed out that the PAR approach to model development offers some important advantages over the more general approach using periodic autoregressive moving-average (PARMA) models.

I have written S functions for the periodic autoregressive modelling methods discussed in my paper. Complete S style documentation for each function is provided. To obtain, e-mail the following message: *send pear from S* to statlib@temper.stat.cmu.edu or use anonymous ftp to connect to fisher.stats.uwo.ca and download the shar archive file, pear.sh, located in the directory pub/pear.

**Key words.** Periodically correlated time series; periodic autoregressive moving-average models; portmanteau test; residual autocorrelation.

## 1. INTRODUCTION

Let $z_t, \quad t = 1, \ldots, N$ be $N$ consecutive observations of a seasonal time series with seasonal period $s$. For simplicity, assume that $N/s = n$ is an integer. In other words, there are $n$ full years of data available. The extension of the results in this paper to the more general case will be seen to be immediate.

The time index parameter, $t$, may be written $t = t(r, m) = (r-1)s + m$, where $r = 1, \ldots, n$ and $m = 1, \ldots, s$. In the case of monthly data $s = 12$ and $r$ and $m$ denote the year and month.

If

$$\mu_m = \mathrm{E}\{z_{t(r,m)}\} \tag{1.1}$$

and

$$\gamma_{\ell,m} = \mathrm{Cov}(z_{t(r,m)}, z_{t(r,m)-\ell}) \tag{1.2}$$

exist and depend only on $\ell$ and $m$, $z_t$ is said to be periodically correlated or periodic stationary (Gladyšev, E.G., 1961). Note that case where $\mu_m$ and $\gamma_{\ell,m}$ do not depend on $m$ reduces to an ordinary covariance stationary time series. Several recent researchers have given methods for testing for periodically correlated time series (Hurd, H.L. and Gerr, N.L., 1991; Vecchia, A.V. and Ballerini, R., 1991). McLeod (1992, to appear) derives a test for detecting periodic correlation in the residuals of fitted seasonal ARIMA models.

The PAR model of order $(p_1, \ldots, p_s)$ may be written,

$$z_{t(r,m)} = \mu_m + \sum_{i=1}^{p_m} \phi_{i,m}(z_{t(r,m)-i} - \mu_{m-i}) + a_{t(r,m)}, \tag{1.3}$$

where $a_{t(r,m)} \sim \mathrm{NID}(0, \sigma_m^2)$. Also, it is understood that $m$ obeys modular arithmetic, for example, $\mu_0 = \mu_s$. The PAR family of models was originally introduced by Thomas and Fiering (1962) for monthly river flow modelling and simulation.

As shown by Troutman (1979), the PAR model may be written in moving-average form

$$z_{t(r,m)} = \mu_m + \sum_{i=0}^{\infty} \psi_{i,m} a_{t(r,m)-i}, \tag{1.4}$$

where the $\psi$'s may be calculated recursively using

$$\psi_{i,m} = \sum_{j=1}^{p_m} \phi_{j,m} \psi_{i-j,m-j}, \quad i \geq 1, \tag{1.5}$$

$\psi_{0,m} = 1$ and $\psi_{i,m} = 0$ if $i < 0$. Then a necessary and sufficient condition for periodic stationarity is (Troutman, 1979)

$$\sum_{i=0}^{\infty} \psi_{i,m}^2 < \infty, \quad m = 1, \ldots, s. \tag{1.6}$$

The PARMA model is a possible extension of the PAR model which includes a moving-average component. Such models have been advocated by Vecchia, A.V. (1985a, 1985b) and others but some drawbacks to their use in actual applications are discussed in §2–5.

## 2. PAR IDENTIFICATION

An illustrative time series, discussed by Vecchia and Ballerini (1991), is the time series of mean monthly flows of the Fraser River at Hope, B.C. from March 1912 to December 1990. This data is available by e-mail by sending the following message, *send fraser-river from datasets*, to statlib@temper.stat.cmu.edu.

The strong seasonal component as well as high seasonal variability of this data are evident in the time series trace plot shown in Figure 1. In practice, logarithms of the mean monthly flows are used to stablize the variance.

[Figure 1 about here]

The presence of periodic correlation in a time series can often be indicated by examination of appropriate scatter plots. Figure 2 compares scatter plots for May vs. June and July vs. August. Notice the very weak correlation present in the May vs. June scatter plot as compared with the strong correlation present in the July vs. August plot. These scatter plots clearly indicate periodic correlation. In less extreme cases, it may be helpful to use the statistical tests described by Hurd and Gerr (1991) and Vecchia and Ballerini (1991).

[Figure 2 about here]

The sample periodic autocorrelation function (PeACF) is given by

$$r_{\ell,m} = \frac{c_{\ell,m}}{\surd\{c_{0,m}c_{0,m-\ell}\}}, \tag{2.1}$$

where

$$c_{\ell,m} = \frac{1}{n}\sum_{r}(z_{t(r,m)} - \hat{\mu}_m)(z_{t(r,m)-\ell} - \hat{\mu}_{m-\ell}), \tag{2.2}$$

where $\hat{\mu}_m = \sum_r z_{t(r,m)}/n$ $n$ is the ceiling of $N/s$ and the summation is over all data values in the sample.

Periodic correlation is evident in the schematic plot of the sample PeACF of the Fraser River time series shown in Figure 3. This schematic plot shows the values $r_{\ell,m}$ for $\ell = 1,\ldots,6$ and $m = 1,\ldots,12$. The lags values, $\ell$, are along the vertical axis. Along the horizontal axis the months, $m$, are represented. Each vertical pair of parallel lines at $\pm 1.96/\surd n$ provide benchmark 5% significance limits valid for white noise.

After the presence of periodic correlation has been detected, a suitable PAR model can be selected either by examining plots of the sample periodic partial autocorrelation (PePACF) or by using an information criterion such as that of Akaike (1974, 1977) or Schwarz (1978). Both the PePACF or information criterion methods can be efficiently implemented using the methods developed by Sakai (1982).

Sakai (1982) extended the celebrated Durbin-Levinson recursion for autoregressive models to PAR models and derived the distribution of the sample PePACF. Let $\hat{\rho}_{\bullet\ell,m}$

denote the sample PePACF for lag $\ell$ and period $m$. Sakai showed that if the correct order is $p_m$ for period $m$, Est.Sd.$(\hat{\rho}_{\bullet\ell,m}) = 1/\sqrt{n}, \quad \ell > p_m$. The order $p_m$ can be identified by finding the lowest lag for which the sample PePACF cuts off.

The BIC criterion (Akaike, 1977; Schwarz, 1978) may be factored to obtained a separate criterion for each period. Thus

$$\mathrm{BIC} = \sum_{m=1}^{s} \mathrm{BIC}_m, \tag{2.3}$$

where

$$\mathrm{BIC}_m = n \ln \hat{\sigma}_m^2 + \ln(n) p_m. \tag{2.4}$$

For the Fraser River data, the BIC method selects a PAR model of order $(1, 1, 1, 3, 2, 1, 1, 3, 1, 1, 1, 1)$. Table I gives the sample PePACF and Figure 3 shows a schematic plot of the sample PePACF. It is seen that the BIC selects a method which is in agreement with that which would be selected by examining the sample PePACF plot. In general, our experience with modelling other monthly river flow time series has been that the BIC and sample PePACF select the same or nearly the same model (see, Noakes, McLeod and Hipel, 1985).

[Table I about here]

[Figure 3 about here]

It should be pointed out that for the more general PARMA model, no satisfactory general method is available for selecting the model orders. The information criterion method is not feasible, particularly when $s \geq 12$, due to the extremely length computations involved.

## 3. PAR ESTIMATION

Let $\boldsymbol{\beta}_m = (\phi_{1,m}, \ldots, \phi_{p_m,m})$ denote the vector of autoregressive parameters for period $m$. Then an asymptotically efficient estimate, $\hat{\boldsymbol{\beta}}_m$, may be obtained by solving Yule-Walker type equations (Pagano, 1978),

$$\sum_{i=1}^{p_m} \hat{\phi}_{i,m} c_{k-i,m-i} = c_{k,m}, \quad k = 1, \ldots, p_m. \tag{3.1}$$

The residual variances may be estimated by

$$\hat{\sigma}_m^2 = c_{0,m} - \hat{\phi}_{1,m} c_{1,m} - \ldots - \hat{\phi}_{p_m,m} c_{p_m,m}, \quad m = 1, \ldots, s. \tag{3.2}$$

Pagano (1978) showed that $\sqrt{n}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})$ is asymptotically normally with mean zero and covariance matrix $\frac{1}{n} I_m^{-1}$, where

$$I_m = \frac{1}{\sigma_m^2} (\gamma_{i-j,m-j}). \tag{3.3}$$

In practice, an estimate, $\hat{I}_m$ of $I_m$ is simply obtained by replacing the $\gamma$'s with $c$'s.

Pagano (1978) also showed that the estimates for different periods are asymptotically uncorrelated or in other words the joint information matrix of $\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_s$ is block diagonal. Pagano (1978) stressed since the PAR model had the property that each period or component could be estimated independently that this could provide a useful approach to certain problems in multivariate autoregression modelling. This theme was taken up and developed further by Newton (1982).

In the PARMA case, in order to obtain efficient estimates, all parameters must be estimated simultaneously including the innovation variances and moreover it is necessary to use a nonlinear optimization technique since the likelihood function is nonlinear. Each evaluation of the likelihood function involves very lengthy computations when $s \geq 12$ (Vecchia, 1985a; Li and Hui, 1988; Jiménez, McLeod and Hipel, 1989).

## 4. PAR DIAGNOSTIC CHECKING

Let $\hat{a}_{t(r,m)}$ denote the residuals from a fitted PAR model and let $\hat{r}_{\ell,m}$ denote the residual autocorrelation for lag $\ell$ and period $m$. Then using the technique of McLeod (1978) or as a specialization of the general multivariate result of Li and McLeod (1981) it can be shown that for any fixed $L \geq 1$ and $\hat{\boldsymbol{r}}_m = (\hat{r}_{1,m}, \ldots, \hat{r}_{L,m})$ that $\sqrt{n}\hat{\boldsymbol{r}}_m$ is asymptotically normal with mean zero and covariance matrix

$$\mathrm{Var}(\boldsymbol{r}_m) = 1_L - X_m I_m^{-1} X_m', \tag{4.1}$$

where $1_L$ denotes the $L$-by-$L$ identity matrix and $X_m$ is the $L$-by-$p_m$ matrix with $(i,j)$ entry $-\psi_{i-j,m-j}\sigma_{m-i}/\sigma_m$. Furthermore, $\sqrt{n}\boldsymbol{r}_m$ and $\sqrt{n}\boldsymbol{r}_{m'}$ are asymptotically uncorrelated when $m \neq m'$. This result means that diagnostic checking can be carried out independently for each month. On the other hand, in the general PARMA case the distribution of the residual autocorrelations is much more complicated and the normalized residual autocorrelations are correlated between periods. Thus, in the PARMA case, the only available diagnostic check would be the global multivariate portmanteau test of Li and McLeod (1978).

The standard deviation of $r_{\ell,m}$ may be estimated from (4.1) by using $\hat{I}_m$ and an estimate of $X_m$ obtained by using the estimated parameters. As pointed out in McLeod (1978) and Ansley and Newbold (1979), model mis-specification may be indicated if the values of $|\hat{r}_{i,m}|$ are too large relative to their estimated standard deviations. In particular, it is often useful to check $\hat{r}_{p_m,m}$. This could be formalized as a statistical test.

To check the validity of this test using the lag $p_m$ value of the residual autocorrelation, a simulation experiment was carried out with PAR models with parameter settings $s = 4, 12; n = 20, 50, 100; p_m = 1, m = 1, \ldots, s$. The same parameter was used for each period so that $\phi_{1,1} = \phi_{1,2} = \ldots = \phi_{1,s} = \phi$, where $\phi = -0.9, -0.8, \ldots, 0.8, 0.9$. The mean empirical significance level, $\hat{\alpha}$, was determined using 1000 simulations for each parameter setting of $s$, $n$ and $\phi$. The values of $\hat{\alpha}$ are displayed in Figure 4. The estimated standard error of $\hat{\alpha}$ is about 0.002. It appears then that in many situations, this test provides a reliable yardstick. Exceptions to this occur only when $\phi$ is close to or equal to 0.0. This suggests the crucial importance of selecting the most parsimonious possible model, that is the model which has the fewest number of parameters but passes the diagnostic checks. Including a parameter which is close to 0.0, when there are not enough data values to get a good estimate of it, seems to cause difficulties.

[Figure 4 about here]

To test simultaneously if all residual autocorrelations at lags $1, 2, \ldots, L$ are equal to zero for a specified period $m$, a portmanteau test (Box and Jenkins, 1976) can be used. The statistic

$$Q_m = n \sum_{\ell=1}^{L} \hat{r}^2_{\ell,m}, \tag{4.2}$$

will be referred to as the Box-Jenkins portmanteau. It follows as indicated in Box and Pierce (1970) that $Q_m$ is asymptotically $\chi^2$-distributed with degrees of freedom $L - p_m$ under the assumption that the model is adequate. Since for the case $s = 1$ all the results in §4 reduce to the standard ARMA case, it would be expected, as in Davies, N., Triggs, C.M. and Newbold, P. (1977) and Ljung and Box (1978) that a modified portmanteau test statistic would improve the small sample properties. It is easily seen that the following exact result holds for the periodic correlations of white noise,

$$\begin{aligned} \mathrm{Var}(r_{\ell,m}) &= \frac{n - \frac{\ell}{s}}{n(n+2)}, \qquad \text{if } \ \ell \equiv 0 \bmod s, \\ &= \frac{n - [\frac{\ell - m + s}{s}]}{n^2}, \qquad \text{otherwise,} \end{aligned} \tag{4.3}$$

where $[\bullet]$ denotes the integer part and

$$r_{\ell,m} = \frac{\sum_r a_{t(r,m)} a_{t(r,m)-\ell}}{\sqrt{\{\sum_r a^2_{t(r,m)} \sum_r a^2_{t(r,m-\ell)}\}}}. \tag{4.4}$$

A suitable modified portmanteau statistic is then

$$\tilde{Q}_{L,m} = \sum_{\ell=1}^{L} \frac{\hat{r}^2_{\ell,m}}{\mathrm{Var}(r_{\ell,m})}. \tag{4.5}$$

Since this modification reduces in the case $s = 1$ to that proposed by Ljung and Box (1978), this may be referred to as the Ljung-Box portmanteau statistic for the periodic autoregression case. It follows using similar approximations as in Ljung and Box (1978, p.300) that

$$\mathrm{E}\{\tilde{Q}_{L,m}\} = L - p_m \tag{4.6}$$

and that

$$\mathrm{E}\{Q_{L,m}\} = n \sum_{\ell=1}^{L} \mathrm{Var}(r_{\ell,m}) - p_m. \tag{4.7}$$

Table II below shows that the degree of underestimation of the asymptotic mean, $L - p_m$, decreases as $s$ increases from 1 and that it is less important for small values of $L$ than for larger values of $L$.

[Table II about here]

The empirical significance level for a nominal 5% portmanteau test was investigated for both the Box-Jenkins statistic $\mathrm{E}\{Q_{L,m}\}$ and the Ljung-Box statistic $\mathrm{E}\{\tilde{Q}_{L,m}\}$. The same parameter settings for the PAR model as in the previous simulation experiment with the lag one residual autocorrelation test were used. The mean empirical significance level, $\hat{\alpha}$, over 1000 simulations, was calculated for each of the 19 values $\phi$ used. Boxplots of these 19 values are displayed in Figure 5. The two outliers at $L = 5$ occur in all four cases when $\phi = -0.9$ and $\phi = 0.9$. Note that Est.Sd.$(\hat{\alpha}) = 0.7\%$. The modification clearly is worthwhile in most cases.

[Figure 5 about here]

## 5. APPLICATION TO FRASER RIVER TIME SERIES

In §2, it was shown that this time series exhibits periodic correlation and a possible model was selected. Both the BIC and PePACF methods suggest a PAR(1,1,1,3,2,1,1,3,1,1,1,1) model. The fitted model and its diagnostic checks are displayed in Table III. The only diagnostic check which is significant at the 5% level is the value of $\tilde{Q}_{15,10}$ which achieves a significance level of about 3%. This does not strongly point to model inadequacy since the level is still not too low and hence the likelihood of a type 1 error exists. This PAR model may be compared to the PARMA model fitted by Vecchia and Ballerini (1991) to the same data. Vecchia and Ballerini fit a PARMA model with a first-order autoregressive and first-order moving-average component for each period. As previously discussed no comprehensive method is available for identifying such PARMA models. It appears that this model was selected without considering possible alternatives. Fewer parameters are required by the PAR model and the goodness-of-fit achieved is at least as good or better. The last row of Table III, shows $100(\hat{\sigma}'_m - \hat{\sigma}_m)/\hat{\sigma}_m$ where $\hat{\sigma}'_m$ denotes the residual standard deviation reported by Vecchia and Ballerini (1991, Table 5). This row may be interpreted as the percentage improvement for each period in using the PAR model. Overall there is a 4.7% average improvement. Due to the lack of the block diagonal structure of the PARMA model, model estimation and diagnostic checking are many orders of magnitude more difficult.

In conclusion, the PAR model is likely to be of more use in applications — at least at present.

[Table III about here]

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TABLE II

Approximate values of $\mathrm{E}\{Q_{L,m}\}$ for $m = 1$, $p_m = 1$.

| $s$ | $L = 5$ | $L = 10$ | $L = 15$ | $L = 20$ | $L = 25$ | $L = 30$ | $L = 35$ |
|---|---|---|---|---|---|---|---|
| 1 | 3.5 | 7.6 | 11.1 | 14.2 | 16.8 | 18.9 | 20.5 |
| 4 | 3.8 | 8.6 | 13.2 | 17.6 | 22.0 | 26.2 | 30.3 |
| 12 | 3.9 | 8.8 | 13.6 | 18.4 | 23.1 | 27.8 | 32.5 |
| $\infty$ | 3.9 | 8.8 | 13.7 | 18.6 | 23.5 | 28.4 | 33.3 |