Necessary and Sufficient Condition for Nonsingular Fisher Information Matrix in ARMA and Fractional ARIMA Models

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It is demonstrated that a necessary and sufficient condition for the Fisher information matrix of a causal and invertible ARMA to be nonsingular is that the model not be redundant, that is, the autoregressive and moving-average polynomials have no roots in common. This result is also extended to fractional ARIMA models.

KEY WORDS: ARMA model redundancy, Asymptotic covariance matrix of ARMA model, Elementary proof, Fractional ARIMA, Parsimony in ARMA models, Test for ARMA model redundancy.

Nowadays ARMA and ARIMA time series models are a standard topic in many statistics courses as well as some courses in econometrics and engineering. Short theoretical exercises or supplements such as the derivation of the Theorems given in this article are often helpful to mathematically inclined students and the elementary proofs given here are suitable for inclusion in most courses where ARMA models are discussed.

Provided the Fisher information is nonsingular, its inverse divided by the length of an observed time series can be used to obtain an estimate of the covariance matrix of the parameter estimates. It is frequently assumed in the time series literature that the Fisher information matrix is nonsingular provided the ARMA model is not redundant. For examples see, Brockwell and Davis (1991, Theorem 8.11.1), and Hannan (1970, Theorem 8, p.403 and pp.413–414). Also, McLeod (1984) and Klein and Mélard (1990) have given convenient algorithms for computing the inverse of Fisher information matrix for ARMA models. Poskitt and Tremayne (1981, p.977) and Wang (1993) have previously

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established that a necessary and sufficient condition for the Fisher information matrix to be nonsingular is that the model not be redundant. But the method presented here is more elementary and is also extended to the case of fractional ARIMA models.

Symbolic computer algorithms in *Mathematica* and numeric algorithms in Splus for the ARMA Fisher information matrix and also its inverse have been archived with *MathSource* and *Statlib* and are also available on my homepage (McLeod, 1998a).

It should be noted that exact model redundancy of the type discussed in this article does not occur with fitted models. Instead if the underlying model is or may be assumed to be redundant, the variance estimates of the parameters are much larger than in a more parsimonious and non-redundant model. Large variances of the parameter estimates are a symptom that the observed information matrix is nearly singular.

The ARMA(p,q) model, which may be written in operator notation as, $\phi(B)z_t = \theta(B)a_t$, where $\phi(B) = 1 - \phi_1 B - \ldots - \phi_p B^p, \theta(B) = 1 - \theta_1 B - \ldots - \theta_q B^q, a_t$ is white noise with variance $\sigma_a^2 > 0$ and B is the backshift operator on t, is said to be not redundant if and only if $\phi(B) = 0$ and $\theta(B) = 0$ have no common roots. The model is assumed causal and invertible which is equivalent to the condition that all roots of $\phi(B)\theta(B) = 0$ are outside the unit disk. To avoid the null case we also assume $\max(p,q) > 0$.

The Fisher information matrix may be written (Box and Jenkins, 1976, p.240), $I(\phi,\theta) = \sigma_a^{-2}\mathrm{E}\{A_t A_t'\}$, where $A_t' = (v_{t-1},\ldots,v_{t-p},u_{t-1},\ldots,u_{t-q})$, $\phi(B)v_{t-1} = -a_{t-1}$, $\theta(B)u_{t-1} = a_{t-1}$ and A_t' denotes the transpose of A_t. Hence,

$$\sigma_a^2 I(\phi,\theta) = \begin{array}{c c} & \begin{array}{cc} p & q \end{array} \\ \begin{array}{c} p \\ q \end{array} & \begin{pmatrix} \gamma_{v,v}(i-j) & \gamma_{v,u}(i-j) \\ \gamma_{u,v}(i-j) & \gamma_{u,u}(i-j) \end{pmatrix} \end{array}$$

and the matrices $(\gamma_{v,v}(i-j))$ and $(\gamma_{u,u}(i-j))$, where, $\gamma_{a,b}(l) = \mathrm{E}(a_t b_{t+l}) \quad (a,b = u,v)$, are positive definite since v_t and u_t are stationary and with positive variance.

Theorem 1. The information matrix $I(\phi,\theta)$ is nonsingular if and only if the model is not redundant.

Proof. If $p = 0$ then the model is not redundant and the information matrix consists of the q-by-q submatrix, $(\gamma_{u,u}(i-j))$ which is positive-definite. Similarly, when $q = 0$.

When $p,q > 0$, $I(\phi,\theta) = \sigma_a^{-2}\mathrm{E}\{A_t A_t'\}$ is nonnegative definite and it is singular if and only if $\ell' I(\phi,\theta)\ell = 0$ for some $p+q$-dimensional vector $\ell \neq 0$. Let $\ell = (\alpha_0,\ldots,\alpha_{p-1},\beta_0,\ldots,\beta_{q-1})$ and note that since the sub-blocks $(\gamma_{v,v}(i-j))$ and $(\gamma_{u,u}(i-j))$ are positive definite, we must have $\alpha_i \neq 0$ for some i as well as $\beta_j \neq 0$ for some j. Since $I(\phi,\theta) = \sigma_a^{-2}\mathrm{E}\{A_t A_t'\}$

it follows that,

$$\mathrm{Var}(\sum_{i=0}^{p-1}\alpha_i v_{t-1-i}+\sum_{j=0}^{q-1}\beta_j u_{t-1-j})=0 \qquad (1.1).$$

Eq. (1.1) is equivalent to the condition that there exists nonzero polynomials $\alpha(B)=\alpha_0+\alpha_1 B+\ldots+\alpha_{p-1}B^{p-1}$ and $\beta(B)=\beta_0+\beta_1 B+\ldots+\beta_{q-1}B^{q-1}$, such that

$$\alpha(B)v_{t-1}+\beta(B)u_{t-1}=0. \qquad (1.2)$$

Eq. (1.2) can now be written

$$\frac{\alpha(B)}{\phi(B)}a_{t-1}-\frac{\beta(B)}{\theta(B)}a_{t-1}=0. \qquad (1.3)$$

Multiplying (1.3) by a_{t-1} and taking expected values, we see that $\alpha(B)/\phi(B)-\beta(B)/\theta(B)=0$ and hence

$$\phi(B)/\theta(B)=\alpha(B)/\beta(B). \qquad (1.4)$$

Since the maximum degrees of the non-zero polynomials $\alpha(B)$ and $\beta(B)$ are respectively $p-1$ and $q-1$ respectively, it follows that the polynomials $\phi(B)$ and $\theta(B)$ contain a common factor and so the model is redundant. Conversely if the model is redundant, eqn. (1.4) holds for some non-zero polynomials $\alpha(B)$ and $\beta(B)$ with maximum degree less than or equal to $p-1$ and $q-1$ respectively. Hence eqn. (1.2) and (1.1) hold and so $I(\phi,\theta)$ is singular. This establishes that lack of model redundancy is a necessary and sufficient condition for the Fisher information matrix, $I(\phi,\theta)$ to be nonsingular and completes the proof.

The extension of this result is immediate to seasonal ARMA models. The result can also be extended to the fractional causal and invertible ARIMA models which are defined by the model equation, $\phi(B)\nabla^d z_t=\theta(B)a_t$, where $\nabla=1-B$ and $\phi(B)$ and $\theta(B)$ are as previously defined and the parameter d can take on any real-value. Diebold and Rudebusch (1989), Beran (1995) and McLeod (1998b) have used this approach for modeling nonstationary data. For this model, Li and McLeod (1986) showed that the Fisher information matrix can be expressed as $I(d,\phi,\theta)=\sigma_a^{-2}\mathrm{E}\{A_t A_t'\}$, where

$$A_t'=(w_{t-1},v_{t-1},\ldots,v_{t-p},u_{t-1},\ldots,u_{t-q}),$$

$$w_{t-1}=-\sum_{j=1}^{\infty}\frac{1}{j}a_{t-j},$$

and v_t and u_t are as defined previously. As for the ARMA case, model redundancy occurs when $\phi(B)$ and $\theta(B)$ share common roots.

Theorem 2. The information matrix $I(d,\phi,\theta)$ in the fractional ARIMA model is nonsingular if and only if the model is not redundant.

Proof. Sufficiency follows immediately from Theorem 1. Next, suppose the model is not redundant. Then singularity of the Fisher information matrix implies there exist $\delta, \alpha_0, \ldots, \alpha_{p-1}, \beta_0, \ldots, \beta_{q-1}$ such that

$$\mathrm{Var}(\delta w_{t-1} + \sum_{i=0}^{p-1} \alpha_i v_{t-1-i} + \sum_{j=0}^{q-1} \beta_j u_{t-1-j}) = 0 \tag{1.5}$$

and furthermore, Theorem 1 implies that $\delta \neq 0$ since the $p+q$ submatrix corresponding to $I(\phi, \theta)$ can not be singular. Since v_{t-1} and u_{t-1} can be expressed as infinite moving average processes with coefficients of $a_{t-1-i}, i = 0, 1, 2, \ldots$ which damp out exponentially fast and since w_{t-1} has coefficients of $a_{t-1-i}, i = 0, 1, 2, \ldots$ which are damping out at a slower rate, it follows that w_{t-1} is not in the closed span comprised of linear combinations of $\{v_{t-1}, \ldots, v_{t-p}, u_{t-1}, \ldots, u_{t-q}\}$. Hence (1.5) can not hold. This establishes Theorem 2.

McLeod (1993) derived a simpler necessary and sufficient condition for ARMA model redundancy. It is given by $|J| = 0$ where $|J|$ denotes the determinant of the matrix,

$$J = \begin{array}{cc} & \begin{array}{cc} p & q \end{array} \\ p+q & \left(\begin{array}{cc} \theta_{i-j} & -\phi_{i-j} \end{array} \right) \end{array},$$

where, $\phi_0 = \theta_0 = -1, \quad \phi_i = 0 \quad (i < 0 \text{ or } i > p), \quad \theta_i = 0 \quad (i < 0 \text{ or } i > q)$ and the (i, j) entry in each partitioned matrix is indicated. For example, it follows that an ARMA(2,1) is redundant if and only if $\phi_2 = \theta_1(\theta_1 - \phi_1)$.

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