FM 3613 — Mathematics of Financial Options

Chapter 5

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1 First-order difference equations

Linear difference equations may be solved in a manner very similar to linear differential equations.

Find a formula for X_k if

$$X_{k+1} = \alpha X_k + \beta$$
$$X_0 = 1,$$

where α and β are fixed constants. This is also known as an initial value problem since the initial value X_0 of the sequence is given. We proceed by analogy—to solve a first-order linear differential equation; we would first divide it into a homogeneous and a particular solution:

$$X_k = H_k + P_k.$$

The homogeneous problem satisfies $H_{k+1} = \alpha H_k$. It is clear that the solution of this homogeneous problem is simply $H_k = \alpha^k H_0$, where H_0 is a constant.

The particular solution is anything that solves

$$P_{k+1} = \alpha P_k + \beta.$$

Let us try a constant $P_k = C$. Then, we have $C = \alpha C + \beta$, or $C(1 - \alpha) = \beta$, provided $\alpha \neq 1$ we obtain

$$P_k = \frac{\beta}{1-\alpha}.$$

Thus, the overall solution is

$$X_k = \alpha^k H_0 + \frac{\beta}{1-\alpha}.$$

To find the constant H_0, we use the initial condition. When $k = 0$, we have:

$$1 = X_0 = \alpha^0 H_0 + \frac{\beta}{1-\alpha}.$$

So,

$$H_0 = 1 - \frac{\beta}{1-\alpha}$$

giving us

$$\begin{aligned} X_k &= \Big(1 - \frac{\beta}{1-\alpha}\Big)\alpha^k + \frac{\beta}{1-\alpha} \\ &= \alpha^k + \frac{\beta}{1-\alpha}(1-\alpha^k). \qquad (1) \end{aligned}$$

It is a good idea to plug this solution into the initial value problem to make sure it works (it does!).

A first-order linear difference equation requires the specification of just one piece of data, be it an initial condition or a final condition. (This requirement for a single piece of data is reminiscent of the ordinary differential equation setting.)

Exercise 1. *Solve the first order difference equation directly by substitution. That is, express* X_k*, in terms of* X_{k-1}*, express* X_{k-1} *on its turn in terms of* X_{k-2} *and so on until you reach* X_0*.*

2 Repaying a loan over time with constant payments

2.1 Repaying with discrete payments

We borrow \$X at time $t = 0$ and repay it by paying \$A at N equal time intervals of length ΔT until at time $N\Delta T$, we owe no money. We pay a constant and fixed interest rate $r\%$ per time interval ΔT.

We want to pay the loan by making constant payments in each time interval.

Let D_k be the amount we owe just after making the k-th payment, that is, at time $k\Delta T$ plus one instant. Then, D_k grows with interest rate r, compounded simply, applied to D_k but is diminished by the payment A at time $(k+1)\Delta T$. Thus,

$$D_{k+1} = (1+r)D_k - A.$$

We know that $D_0 = X$ and that $D_N = 0$. So, we now have a difference equation with initial and final values specified.

SHOW THE STUDENTS HOW TO FIGURE OUT THE MORTGAGE PAYMENTS WITH A SPREADSHEET.

It is often helpful to remove all the units before getting started by, for instance, dividing out a term. Let us remove the currency units from this equation and non-dimensionalize it by letting

$$d_k := \frac{D_k}{X}, \quad a := \frac{A}{X}.$$

Then our system becomes

$$\begin{aligned} d_{k+1} &= (1+r)d_k - a \text{ and} \\ d_0 &= 1, \\ d_N &= 0 \end{aligned}$$

Dividing by X also removes one of the variables by making it clear that X just scales things: all else being equal, the payments on a million dollar loan are simply 1000 times bigger than the payments on a thousand dollar loan.

We have both an initial and a final value. Is the problem then over specified? Yes, if N, r, and a are all known. Otherwise, we can control at least one of these variables. Typically, r is outside our control, but both N and a can be manipulated.

Now, we use equation (1) with $\alpha = 1 + r$ and $\beta = -a$ to obtain

$$\begin{aligned} d_k &= (1+r)^k - \frac{a}{1-(1+r)}\big(1-(1+r)^k\big) \\ &= (1+r)^k + \frac{a}{r}\big(1-(1+r)^k\big) \\ &= \Big(1-\frac{a}{r}\Big)(1+r)^k + \frac{a}{r}. \end{aligned}$$

Now, we use the final condition $d_N = 0$ to get the interrelation between the parameters:

$$\Big(1-\frac{a}{r}\Big)(1+r)^N + \frac{a}{r} = 0$$

or, solving for a, we get

$$a = \frac{r(1+r)^N}{(1+r)^N - 1}$$

and since $A = aX$ we obtain

$$A = \frac{rX(1+r)^N}{(1+r)^N - 1}. \tag{2}$$

Of what use is this expression? Why is it better than coding the recursion relation into your spreadsheet and tuning A until the principal vanishes at the end? If our objective is to obtain the value of our car loan payment, then the formula for A is not really that much of an advantage over the spreadsheet. However, it allows us to use powerful techniques to extract insight about this problem.

We consider two special cases now.

Case 1. What happens if the number of payment periods N approaches infinity? That is, we end up with a perpetual loan. Then

$$A = \lim_{N\to\infty} \frac{rX(1+r)^N}{(1+r)^N - 1} = rX \lim_{N\to\infty} \frac{(1+r)^N}{(1+r)^N - 1} = rX,$$

since $r > 0$. This is intuitive, as in this case, the loan is never repaid, which is merely maintained at a constant level.

Case 2. What happens if the number of payment periods $N = 1$? Then

$$A = \frac{rX(1+r)^1}{(1+r)^1 - 1} = (1+r)X.$$

This is also intuitive, since if there is just a single period, we must repay the principal X and the accrued interest rX.

Case 3. What happens if the interest rate $r = 0$? If we plug $r = 0$ into formula (2), then both the numerator and the denominator will become zero leading to the nonsense $0/0$. What we need to find is the limit of formula (2) as r approaches zero. For that goal, divide the numerator and the denominator by r:

$$\lim_{r\to 0} \frac{rX(1+r)^N}{(1+r)^N - 1} = \lim_{r\to 0} \frac{X(1+r)^N}{\big((1+r)^N - 1\big)/r}.$$

Now consider the function $f(x) = x^N$ and recall the definition of a derivative

$$N = f'(1) = \lim_{r\to 0} \frac{f(1+r) - f(1)}{r} = \lim_{r\to 0} \frac{(1+r)^N - 1}{r}.$$

So, we find

$$\lim_{r\to 0} \frac{X(1+r)^N}{\big((1+r)^N - 1\big)/r} = \frac{X}{N},$$

as one expects.

Exercise 2. *Suppose you cannot afford to pay more than \$A in each time period. At least how many payments do you need to make to repay the loan if the interest rate every time period is r?*

Exercise 3. *If you decide to increase the number of payments N, will A increase or decrease, keeping the interest rate fixed? Does this make sense?*

Exercise 4. *If the interest rate r increase, will this increase the payment A, keeping the number of payments N fixed? Does this make sense?*

Exercise 5. *Suppose a loan of size \$X can be repaid in N instalments of size \$A. Suppose you take a loan of size $X+Y$ for some $Y \geq 0$. How many more payments do you need to make in order to repay the loan if you can only afford to make the same payments. The interest rate in each term is r.*

2.1.1 Approximating the payment

Let us use this to expand equation (2) in a Taylor series about $r = 0$. This is easily done with MAPLE with the command

$$\text{simplify}\Big(\text{taylor}\Big(\tfrac{rX(1+r)^N}{(1+r)^N-1}, r = 0, 5\Big)\Big).$$

The result is

$$A = \frac{X}{N}\Big(1 + (1/2)(N+1)r + (1/12)(N^2-1)r^2 - (1/24)(N^2-1)r^3\Big) + O(r^4)$$

In fact, one can show the inequalities

$$\frac{X}{N}\Big(1 + (1/2)(N+1)r\Big) \leq A \leq \frac{X}{N}\Big(1 + (1/2)(N+1)r + (1/12)(N^2-1)r^2\Big) \tag{3}$$

Note that

$$\frac{X}{N} + (1/2)rX \leq \frac{X}{N}\Big(1 + (1/2)(N+1)r\Big),$$

so a crude rule of thumb for the payments A is to divide the initial principle by N and add half of the interest on the full principle. This rule of thumb is an underestimation of A.

Exercise 6. Show that inequalities (3) hold.

2.1.2 Repaying the loan "quickly"

First, what does "quickly" mean? In this setting, "quickly" means "before too much interest accrues."

Definition 7. We say that the loan is repaid *quickly* when the total amount of interest paid is not more than the loan amount X.

Exercise 8. *Show that the total amount of (undiscounted) interest paid over the lifetime of the loan is*

$$-X + \frac{rNX(1+r)^N}{(1+r)^N - 1}.$$

Example 9. Using the first-order Taylor approximation to $(1+r)^N$, we can find an upper bound on the number of payments, N, so that the loan is repaid quickly. Indeed, start by recalling the binomial expansion

$$(1+r)^N = \sum_{k=0}^{N} \binom{N}{k} r^k = 1 + Nr + \frac{N(N-1)}{2} r^2 + o(r^2).$$

Thus, using the fact that the function $x \mapsto x/(x-1)$ is decreasing over $(1, \infty)$, we get an upper bound for the total (undiscounted) interest paid (see Exercise 8)

$$-X + \frac{rNX(1+r)^N}{(1+r)^N - 1} \leq -X + \frac{rNX(1+Nr)}{(1+Nr) - 1} = rNX.$$

Thus, if $N < 1/r$ we get that $rXN < X$ and the loan is repaid quickly. This is a first order upper bound for N that guarantees a quick repayment of the loan.

Exercise 10. Suppose that the interest rate $r \in [0, 1]$. Using second-order Taylor approximation to $(1+r)^N$, find an upper bound on the number of payments, N, so that the loan is repaid quickly. Is this upper bound bigger or smaller than the one you found in Exercise 9? What does this mean?

2.2 Repaying continuously

2.2.1 Continuously compounded interest rates

The first thing to remember is that interest rates are always quoted per year. Say CIBC offers deposits that will earn interest rate j at the end of a year. That is, if you deposit X dollars now, after one year CIBC will return $(1+j)X$ dollars to you. Suppose that RBC offers deposits that will pay interest, call it r, after six months, compound it (meaning it is added to the principle), and then pay the same interest r, on the compounded amount after another six months. How much should the interest rate r be, so that there is no arbitrage between the two banks? (This means that no one can make sure profit by exploiting the interest rates of the two banks.) If you deposit X dollars in the two banks, they should grow to the same amount after one year:

$$(1+j)X = (1+r)(1+r)X.$$

If this equality does not hold, say if we have $(1+j)X > (1+r)(1+r)X$ then one can take a loan of X dollars from RBC and deposit it in CIBC. After one year, CIBC will return them $(1+j)X$ which will be more than enough to pay the loan, which grew to $(1+r)^2X$, to RBC, thus pocketing the difference $(1+j)X - (1+r)^2X = ((1+j)-(1+r)^2)X$. By taking bigger and bigger loans X, one can make an unlimited fortune.

Solving for r, we find that interest rate *over the period of six months* should be

$$r = (1+j)^{1/2} - 1.$$

Since there are two 6-month periods in a year, and the interest rate at RBC has to be quoted per year, we define

$$j_2 := 2r = 2((1+j)^{1/2} - 1)$$

and say that the interest rate is j_2 *per year compounded semi-annually.*

In general, denote by j_n the interest rate per year compounded n times, at n equally spaced time intervals. By definition, the interest rate over each period is

$$r := \frac{j_n}{n}.$$

The connection between j_n and j_m is

$$\left(1+\frac{j_n}{n}\right)^n = \left(1+\frac{j_m}{m}\right)^m. \tag{4}$$

What happens to j_n when n gets larger and larger? Well, we always must have

$$j_n = n((1+j)^{1/n} - 1),$$

since j is just another notation for j_1. The answer is in the next exercise.

Exercise 11. *Show that for any $a > 0$ we have*

$$\lim_{n\to\infty} n(a^{1/n} - 1) = \log a.$$

The exercise shows that

$$\lim_{n\to\infty} j_n = \log(1+j).$$

Denote that quantity by

$$j_\infty := \log(1+j).$$

We say that j_∞ is the *interest rate per year compounded continuously* or we just say that j_∞ is the *continuously compounded interest rate* or the *force of interest.* This gives us a connection between j_∞ and $j = j_1$. How about between j_∞ and j_m? Notice that from (4) we have

$$j_n = n\left(\left(\left(1+\frac{j_m}{m}\right)^m\right)^{1/n} - 1\right).$$

Thus, taking the limit as n goes to infinity and using the exercise, we get

$$j_\infty = \log\left(\left(1+\frac{j_m}{m}\right)^m\right) = m\log\left(1+\frac{j_m}{m}\right).$$

Conversely, solving for j_m, we have

$$j_m = m\Big(e^{\frac{j_\infty}{m}} - 1\Big)$$

or, alternatively, the interest rate per period (the period is of length $1/m$) is

$$r = \frac{j_m}{m} = e^{j_\infty \frac{1}{m}} - 1. \tag{5}$$

The beauty of continuously compounded interest rate j_∞ is that formula (5) holds not just for periods of length $1/m$ but for periods of *any* length ΔT. Thus, if we know the continuously compounded interest rate j_∞ then the interest rate, r, over a period of length ΔT is

$$r = e^{j_\infty \Delta T} - 1. \tag{6}$$

Exercise 12. *Show that formula (6) holds for any period of length ΔT. (Hint: We showed the formula for periods $\Delta T = 1/m$ for integer $m > 0$. Next show the formula for periods $\Delta T = n/m$ where $n, m > 0$ are integers. Finally, for arbitrary ΔT, approximate the real number ΔT by rational numbers n/m and take a limit.)*

From now on, the continuously compounded interest rate j_∞ will be denoted by R and called the *force of interest*.

We say that the function $B(t)$ is the *rate* at which the loan is repaid if the money that is paid over a time period $[t, t + \Delta T]$ is

$$\int_t^{t+\Delta T} B(s)\, ds.$$

For example if a loan is repaid at a rate B (a constant) then in every time period of length ΔT, we pay back $B\Delta T$ dollars.

2.2.2 Take limit as $\Delta T \to 0$ first, solve differential equation second

We take a loan of size X now and want to repay it *continuously* between now, time 0, and time T. The interest on the loan is compounded continuously with force R and we repay the loan continuously at rate B.

Our goal is to find B so that the loan is repaid by time T.

Let $D(t)$ be the size of the loan at time $t \in [0, T]$. Since T is the time when the loan is completely repaid, we have the boundary conditions

$$D(0) = X \text{ and } D(T) = 0.$$

To create a differential equation for $D(t)$ consider the interval $[t, t + \Delta T]$, for a small ΔT, and consider how the function $D(t)$ changes over this interval:

$$D(t + \Delta T) = (1 + r)D(t) - A,$$

where r is the interest rate over the period $[t, t + \Delta T]$, of length ΔT, and A is the amount paid over the period. From the previous section we have that

$$1 + r = e^{R\Delta T} \text{ and } A = \int_t^{t+\Delta T} B\, ds = B\Delta T.$$

Thus

$$D(t + \Delta T) = e^{R\Delta T} D(t) - B\Delta T$$

holds for every time period ΔT. For small time periods ΔT, by the first order Taylor expansion, we have

$$e^{R\Delta T} = 1 + R\Delta T + o(\Delta T).$$

Substituting above and regrouping gives

$$D(t + \Delta T) - D(t) = (R\Delta T + o(\Delta T))D(t) - B\Delta T.$$

Diviging by ΔT and letting it approach 0, we arrive at the differential equation:

$$D'(t) = RD(t) - B.$$

This can be solved by using a homogeneous and particular solution. The homogeneous part is

$$D'(t) = RD(t)$$

with solution $D_h(t) = Ce^{Rt}$, where C is some constant. We try the constant function $D_p(t) := K$ to see if it can solve the non-homogeneous equation. In order to do that, K must satisfy

$$0 = D_p'(t) = RD_p(t) - B = RK - B$$

or $K = B/R$. Adding the homogeneous and particular solutions gives a solution

$$D(t) = Ce^{Rt} + \frac{B}{R}.$$

Using the condition $D(0) = X$ we find C:

$$X = D(0) = C + \frac{B}{R}$$

or $C = X - B/R$. So, finally

$$D(t) = \left(X - \frac{B}{R}\right)e^{Rt} + \frac{B}{R} = Xe^{Rt} + \frac{B}{R}(1 - e^{Rt}).$$

Remember that our goal is to find B, the rate of repayment. To do that, we use $D(T) = 0$:

$$Xe^{RT} + \frac{B}{R}(1 - e^{RT}) = 0$$

or

$$\frac{B}{R} = \frac{Xe^{RT}}{e^{RT} - 1} \quad \text{or} \quad B = \frac{RXe^{RT}}{e^{RT} - 1}.$$

That is the solution we get for continuous repayment when we take the limit first and solve the differential equation second.

2.2.3 Solve difference equation first, take limit as $\Delta T \to 0$ second

We take a loan of size X now and want to repay it *continuously* between now, time 0, and time T. The interest on the loan is compounded continuously with force R and we repay the loan continuously at rate B.

Our goal is to find B so that the loan is repaid by time T.

Let $D(t)$ be the size of the loan at time $t \in [0, T]$. Since T is the time when the loan is completely repaid, we have the boundary conditions

$$D(0) = X \text{ and } D(T) = 0.$$

Divide the interval $[0, T]$ into N equal parts of size

$$\Delta T := \frac{T}{N}. \tag{7}$$

Our strategy now is to create a recursive relationship relating $D(0 + (k+1)\Delta T)$ with $D(0 + k\Delta T)$, solve the recursive relationship to find B, and then let ΔT approach 0.

Let

$$D_k := D(0 + k\Delta T) \text{ for } k = 0, 1, \ldots, N.$$

Clearly, $D_0 = X$ and $D_N = 0$. As before, these quantities must satisfy

$$D_{k+1} = (1 + r)D_k - A \text{ for all } k = 0, 1, \ldots, N - 1,$$

where r is the interest rate over the period $[k\Delta T, (k + 1)\Delta T]$, of length ΔT, and A is the amount paid over the same period. That is, we have that

$$r = e^{R\Delta T} - 1 \text{ and } A = B\Delta T.$$

We know that the solution of the recursive relationship is

$$A = \frac{rX(1 + r)^N}{(1 + r)^N - 1} \tag{8}$$

or, substituting A and r with their equals:

$$B\Delta T = \frac{(e^{R\Delta T} - 1)X(e^{R\Delta T})^N}{(e^{R\Delta T})^N - 1}.$$

Thus, dividing both sides by ΔT, we get

$$B = \frac{e^{R\Delta T} - 1}{\Delta T}\frac{Xe^{RT}}{e^{RT} - 1}, \tag{9}$$

where we also used that

$$(e^{R\Delta T})^N = e^{R\Delta TN} = e^{RT},$$

according to (17). Now, using the first order Taylor expansion of $e^{R\Delta T} = 1 + R\Delta T + o(\Delta T)$ we find

$$\frac{e^{R\Delta T} - 1}{\Delta T} = \frac{(1 + R\Delta T + o(\Delta T)) - 1}{\Delta T} = \frac{R\Delta T + o(\Delta T)}{\Delta T} = R + \frac{o(\Delta T)}{\Delta T} \to R \text{ as } \Delta T \to 0.$$

Thus, taking limit at $\Delta T \to 0$ in (9), we get

$$B = \frac{RXe^{RT}}{e^{RT} - 1} \tag{10}$$

That is the solution-first-limit-second answer, and it agrees with our earlier solution, which was the answer in the limit-first-solution-second method. Excellent!

Exercise 13. *What happens with the rate of payment B when T increases? Why?*

Exercise 14. *What happens with the rate of payment B when R increases? Why?*

Exercise 15. *What happens with the rate of payment B when T increases and R decreases in such a way that RT stays constant? Why?*

Exercise 16. *When we repay the loan with discrete payments, we pay amount A, every time period of length ΔT. When we repay the loan continuously, we pay amount $B\Delta T$, every time period of length ΔT. In which case do we pay more over a time period of length ΔT? Does the answer make financial cense?*

3 Repaying a loan over time with increasing payments

3.1 The discrete case

Consider a mortgage with principal value X. This mortgage is repaid in N payments made every ΔT years. The rate of interest is constant at $r\%$ per ΔT period. However, the payments are not equal. Instead, they are an affine function of time, with the k-th payment being

$$A_k = A_0 + (k-1)B.$$

Considering A_0 as fixed, at time ΔT we make a payment of size A_0, at time $2\Delta T$ we make a payment of size $A_0 + B$, and so on, at time $N\Delta T$ we make a payment of size $A_0 + (N-1)B$

Find B required to exactly repay the mortgage at the end of the N-th payment. Thus, B will be a function of X, r, N, and A_0.

Let D_k denote the balance owing immediately after the k-th payment. We can now write

$$D_{k+1} = (1+r)D_k - A_{k+1},$$

equivalently

$$D_{k+1} = (1+r)D_k - (A_0 + kB), \tag{11}$$

with boundary conditions

$$D_0 = X \text{ and } D_N = 0.$$

After dividing by the principal amount X, we arrive at

$$d_{k+1} = (1+r)d_k - a_{k+1}, \tag{12}$$

where $a_{k+1} = a_0 + kb$ where $a_0 = A_0/X$ and $b = B/X$. The boundary conditions become

$$d_0 = 1 \text{ and } d_N = 0.$$

To solve this difference equation, we begin by finding the homogenous solution that we denote by H_k. The homogeneous problem is

$$H_{k+1} = (1+r)H_k$$

yielding the solution $H_k = c(1+r)^k$ for some constant c.

In order to find a solution of the general difference equation, we try the trick to allow the coefficient c to vary with k. That is, we are going to try a solution of (12) of the following form

$$d_k = c_k(1+r)^k.$$

To find c_k we insert into (12):

$$c_{k+1}(1+r)^{k+1} = c_k(1+r)^k(1+r) - a_{k+1}.$$

This leads to

$$(c_{k+1} - c_k)(1+r)^{k+1} = -a_{k+1}$$

with boundary conditions $c_0 = d_0/(1+r)^0 = 1$ and $c_N = d_N/(1+r)^N = 0$.

Now, let $e_k := c_{k+1} - c_k$ we obtain:

$$e_k = -\frac{a_{k+1}}{(1+r)^{k+1}}$$

and

$$c_k = c_0 + e_0 + \cdots + e_{k-1}$$

But $c_0 = 1$ so we can write

$$\begin{aligned} c_k &= 1 + \sum_{j=0}^{k-1}\Big(-\frac{a_{j+1}}{(1+r)^{j+1}}\Big) = 1 - \sum_{j=1}^{k}\frac{a_j}{(1+r)^j} \\ &= 1 - \sum_{j=1}^{k}\frac{a_0 + (j-1)b}{(1+r)^j}, \end{aligned}$$

where we recalled that $a_j = a_0 + (j-1)b$. Thus, our solution is

$$d_k = \Big(1 - \sum_{j=1}^{k}\frac{a_0+(j-1)b}{(1+r)^j}\Big)(1+r)^k. \tag{13}$$

We can check to ensure that $d_0 = 1$ (sum from $j = 1$ to $j = 0$ is empty).

Our next task is to use the boundary condition $d_N = 0$ in order to solve for b.

First, we recall that

$$\begin{aligned} \sum_{j=1}^{N}\frac{1}{(1+r)^j} &= \frac{1 - 1/(1+r)^{N+1}}{1 - 1/(1+r)} - 1 = \frac{(1+r)^{N+1} - 1}{r(1+r)^N} - 1 \\ &= \frac{(1+r)^N - 1}{r(1+r)^N}. \end{aligned}$$

Second, we have the following lemma.

Lemma 17. *The following identity holds*

$$\sum_{j=1}^{N} \frac{(j-1)}{(1+r)^j} = -\frac{Nr-(1+r)^N+1}{r^2(1+r)^N}.$$

Proof. Begin by noting that on the one hand we have

$$\frac{d}{dr}\Big(\sum_{j=1}^{N} \frac{1}{(1+r)^{j-1}}\Big) = -\sum_{j=1}^{N} \frac{(j-1)}{(1+r)^j}.$$

On the other hand

$$\sum_{j=1}^{N} \frac{1}{(1+r)^{j-1}} = \frac{1-1/(1+r)^N}{1-1/(1+r)} = \frac{(1+r)^N-1}{r(1+r)^{N-1}}.$$

Differentiating both sides of the last equality with respect to r, and after some simplifications, we obtain

$$\sum_{j=1}^{N} \frac{(j-1)}{(1+r)^j} = -\frac{Nr-(1+r)^N+1}{r^2(1+r)^N}.$$

That is what we needed to show. $\square$

We now return to equation (13) with $k=N$, use the boundary condition $d_N=0$, and cancel $(1+r)^N$ from the right-hand side:

$$\begin{aligned} 0 &= 1-\sum_{j=1}^{N}\frac{a_0+(j-1)b}{(1+r)^j} = 1-a_0\Big(\sum_{j=1}^{N}\frac{1}{(1+r)^j}\Big)-b\Big(\sum_{j=1}^{N}\frac{(j-1)}{(1+r)^j}\Big) \\ &= 1-a_0\Big(\frac{(1+r)^N-1}{r(1+r)^N}\Big)+b\Big(\frac{Nr-(1+r)^N+1}{r^2(1+r)^N}\Big). \end{aligned}$$

Multiplying both sides by $r^2(1+r)^N$ gives

$$\begin{aligned} 0 &= r^2(1+r)^N - ra_0((1+r)^N-1)+b(Nr-(1+r)^N+1), \\ &= r^2(1+r)^N - ra_0(1+r)^N + a_0r + b(Nr-(1+r)^N+1). \end{aligned}$$

Finally, we solve for b:

$$b = \frac{r(a_0-r)(1+r)^N-a_0r}{Nr-(1+r)^N+1}.$$

Going back to the original B and A_0 we find

$$B = \frac{r(A_0-rX)(1+r)^N-A_0r}{Nr-(1+r)^N+1}. \tag{14}$$

Exercise 18. *Use the result above to consider two special cases. For both special cases, use the numerical values $N = 120$, $\Delta T = 1/12$ year, $X = \$100,000$, and $r = 0.5\%$ (recall, per period!). For each special case, find the corresponding B and plot the balance owing as a function of time.*

i. $A_0 = 0$;

ii. $A_0 = rX$ *(just enough to cover the interest).*

Exercise 19. *Consider a mortgage with principal value X. This mortgage is repaid in N payments made every ΔT years. The rate of interest is constant at $r\%$ per ΔT period. However, the payments are not equal. Instead, they are an affine function of time, with the k-th payment being*

$$A_k = A_0 + kB.$$

At time ΔT we make a payment of size $A_0 + B$, at time $2\Delta T$ we make a payment of size $A_0 + 2B$, and so on, at time $N\Delta T$ we make a payment of size $A_0 + NB$.

Find B required to exactly repay the mortgage at the end of the N-th payment.

Note that Exercise 19 is precisely the case done in class. But also, it is an easy corollary of the discrete case considered above. Indeed, we have

$$A_k = A_0 + kB = (A_0 + B) + (k-1)B.$$

Thus, all we have to do is replace A_0 in (14) by $A_0 + B$ and then solve for B:

$$B = \frac{r(A_0 + B - rX)(1+r)^N - (A_0 + B)r}{Nr - (1+r)^N + 1}$$

and solving for B gives

$$B = \frac{r(A_0 - Xr)(1+r)^N - A_0 r}{Nr - (1+r)^{N+1} + (1+r)}.$$

3.2 The continuous case

3.2.1 Take limit as $\Delta T \to 0$ first, solve differential equation second

We take a loan of size X now and want to repay it *continuously* between now, time 0, and time T. The interest on the loan is compounded continuously with force R and we repay the loan continuously at rate $A_0 + Bt$.

Our goal is to find B so that the load is repaid by time T.

Let $D(t)$ be the size of the loan at time $t \in [0, T]$. Let R be the force of interest and let the repayment rate at time t be $A_0 + Bt$. Since T is the time when the loan is completely repaid, we have the boundary conditions

$$D(0) = X \text{ and } D(T) = 0.$$

To create a differential equation for $D(t)$ consider the interval $[t, t + \Delta T]$, for a small ΔT, and consider how the function $D(t)$ changes over this interval:

$$D(t + \Delta T) = (1 + r)D(t) - A(t).$$

where r is the interest rate over the period $[t, t + \Delta T]$, of length ΔT, and $A(t)$ is the amount paid over the period. We have that

$$1 + r = e^{R\Delta T} \text{ and } A(t) = \int_t^{t+\Delta T} (A_0 + Bs)\, ds.$$

Thus, we have

$$D(t + \Delta T) = e^{R\Delta T} D(t) - \int_t^{t+\Delta T} (A_0 + Bs)\, ds.$$

Divide by ΔT and let it approach zero. First note that

$$\begin{aligned}
\frac{1}{\Delta T} \int_t^{t+\Delta T} (A_0 + Bs)\, ds &= \frac{1}{\Delta T}\Big(A_0 \Delta T + \frac{B}{2}\big((t + \Delta T)^2 - t^2\big)\Big) \\
&= \frac{1}{\Delta T}\Big(A_0 \Delta T + \frac{B}{2}\big((2t\Delta T + (\Delta T)^2\big)\Big) \\
&\to A_0 + Bt \text{ as } \Delta T \text{ approaches } 0.
\end{aligned}$$

Substituting above leads to the differential equation

$$D'(t) = RD(t) - (A_0 + Bt).$$

with boundary conditions $D(0) = X$ and $D(T) = 0$. Divide both sides by X and let $d(t) := D(t)/X$, $a_0 := A_0/X$, and $b := B/X$:

$$d'(t) = Rd(t) - (a_0 + bt). \tag{15}$$

the boundary conditions are $d(0) = 1$ and $d(T) = 0$. To solve the above ODE, we first find a homogeneous solution $d_h(t)$ to the homogeneous equation $d'(t) = Rd(t)$, that is, $d_h(t) = e^{Rt}$. Now, let us try looking for a solution of (15) of the form

$$d(t) = c(t)d_h(t).$$

Differentiating this and plugging in the original ODE, we arrive at

$$c'(t) = -(a_0 + bt)e^{-Rt}$$

with $c(0) = 1$. (Note that $d(0) = 1$ implies $c(0) = 1$.) The latter is a very easy separable differential equation with the following solution:

$$c(t) = -\int (a_0 + bt)e^{-Rt}\, dt + \lambda = \frac{a_0}{R}e^{-Rt} + b\Big(\frac{t}{R} + \frac{1}{R^2}\Big)e^{-Rt} + \lambda.$$

The innitial condition $c(0) = 1$ yields $\lambda = 1 - a_0/R - b/R^2$. Therefore, we have

$$c(t) = \frac{a_0}{R}e^{-Rt} + b\Big(\frac{t}{R} + \frac{1}{R^2}\Big)e^{-Rt} + 1 - \frac{a_0}{R} - \frac{b}{R^2}.$$

Substituting this answer in $d(t) = c(t)d_h(t)$, gives

$$d(t) = \Big(1 - \frac{a_0}{R} - \frac{b}{R^2}\Big)e^{Rt} + \frac{a_0}{R} + b\Big(\frac{t}{R} + \frac{1}{R^2}\Big).$$

To solve for b, we now use the condition $d(T) = 0$ to get

$$b = \frac{R(a_0 - R)e^{RT} - a_0 R}{RT - e^{RT} + 1}.$$

Going back to the original B and A_0 we find

$$B = \frac{R(A_0 - RX)e^{RT} - A_0 R}{RT - e^{RT} + 1}. \tag{16}$$

3.2.2 Solve difference equation first, take limit as $\Delta T \to 0$ second

We take a loan of size X now and want to repay it *continuously* between now, time 0, and time T. The interest on the loan is compounded continuously with force R and we repay the loan continuously at rate B.

Our goal is to find B so that the loan is repaid by time T.

Let $D(t)$ be the size of the loan at time $t \in [0, T]$. Since T is the time when the loan is completely repaid, we have the boundary conditions

$$D(0) = X \text{ and } D(T) = 0.$$

Divide the interval $[0, T]$ into N equal parts of size

$$\Delta T := \frac{T}{N}. \tag{17}$$

Our strategy now is to create a recursive relationship relating $D(0 + (k+1)\Delta T)$ with $D(0 + k\Delta T)$, solve the recursive relationship to find B, and then let ΔT approach 0.

Let

$$D_k := D(0 + k\Delta T) \text{ for } k = 0, 1, \ldots, N.$$

Clearly, $D_0 = X$ and $D_N = 0$. As before, these quantities must satisfy

$$D_{k+1} = (1 + r)D_k - A_{k+1} \text{ for all } k = 0, 1, \ldots, N - 1, \tag{18}$$

where r is the interest rate over the period $[k\Delta T, (k+1)\Delta T]$, of length ΔT, and A_{k+1} is the amount paid over the same period. That is, we have that

$$r = e^{R\Delta T} - 1$$

and

$$\begin{aligned} A_{k+1} &:= \int_{k\Delta T}^{(k+1)\Delta T} (A_0 + Bs)\, ds = \left(A_0 s + \frac{B}{2}s^2\right)\Big|_{s=k\Delta T}^{(k+1)\Delta T} = A_0\Delta T + \frac{B}{2}\big(((k+1)\Delta T)^2 - (k\Delta T)^2\big) \\ &= A_0\Delta T + \frac{B}{2}(\Delta T)^2 + kB(\Delta T)^2. \end{aligned}$$

Exercise 20. *Finish the argument above in order to obtain a formula for B. Show that when ΔT converges to 0, the formula reduces to (16).*