FM 3613 — Mathematics of Financial Options

Chapter 9 - Tranching and Collateralized Debt Obligations

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1 Collateralized debt obligations

Investors can be divided into two broad classes.

Some seek high degrees of safety. They might be life insurers who have sold life annuities and who need to make sure they have the cash on hand to pay their annuitants each month. Such investors are interested in bonds with low probability of default, and are willing to accept rather low coupons in exchange. In addition, certain federally regulated entities such as life insurance companies are forbidden to purchase debt higher than a certain risk.

Other investors seek high returns, and are willing to take risks to get it.

Demand for bonds of intermediate risk is fairly small. The coupon results results in Chapter 7

$$c(p) = \frac{r + p(1-R)}{1 - p(1-R)}$$

suggest that bonds with a probability of default of say 2% per annum should not pay all that much more coupon than a risk-free bond

$$r = c(0) \approx c(0.02) \approx r + 0.02(1+r)(1-R).$$

But simulation shows that even fairly large portfolios of these bonds can have, if times are bad, serious value fluctuations.

It would be great to make a machine that converted middle of the road risk factor bonds into some low-risk bonds and some very high return (if also very high-risk) bonds. The machine that does that exists and it is called the *collateralized debt obligation product*.

The best way to understand this product is to begin with a simple example. Suppose that there are available on the market two bonds. These are very simple bonds. At time $t = 0$ they cost $\$X$, and at time $t = 1$ they repay either $\$X + cX$, with probability p, or 0, with probability $1 - p$. (That is, the recovery rate R is assumed to be zero for these bonds.) Suppose that these two bonds default entirely independent of one another.

Suppose these bonds are risky enough not to be of investment grade. How can we construct an investment grade bond from them?

What we do is purchase the two bonds described above and put them into a special legal structure which guards them from the rest of the world. The cash flows arising from this new structure are distributed between investors in a special way.

We “slice” our portfolio into two equal parts called “tranches” (French for ”slice”). That is, we create two new fixed income securities—one senior and one junior. The seniority refers to the order in which cash flows accrue to investors in the event of a default. The junior “tranche” is only paid its first dollar once the senior tranche has been paid in full.

If none of the bonds defaults at time $t = 1$, then an amount of $2(X + cX)$ is distributed between the senior and the junior tranches. Say, tranche A gets $X + c_A X$ and tranche B gets $X + c_B X$, where we must have $c_A + c_B = 2c$.

If exactly one of the constituent bonds defaults at time $t = 1$, then an amount of $(X + cX)$ is distributed between the senior and the junior tranches. The senior tranche A receives $X + c_A X$ again, while the junior tranche B gets whatever is left.

Only in the devastating and, given the assumption of independence, rather unlikely event of both constituent bonds defaulting then both tranches get 0.

Thus, the senior tranche behaves just like a bond with a probability of default of $p^2 < p$, while the junior tranche is more likely to default than either of the constituent bonds considered alone. That is, the senior tranche is less risky and the junior is more risky. To reflect this, it must be that the coupon paid to the senior tranche holders be smaller than the coupon paid to their junior colleagues. The size of this coupon is what we will calculate next.

2 Tranched portfolios

Suppose a portfolio is made which contains two bonds with identical default probability p and recovery rate R but whose defaults are uncorrelated. For now we assume, for simplicity's sake, that $R = 0$ and that the bonds make just one coupon payment, at maturity. We further suppose that these bonds are sold at par, that is, the coupon rate c is given by

$$c = \frac{r + p}{1 - p}.$$

An investor purchases equal amounts of each bond and creates two new products by tranching. The products are constructed by creating a new legal entity whose only assets are the two bonds. This asset is split into two parts with equal face value: tranche A and tranche B. These two tranches are not, however, identical: tranche A is senior to tranche B. This means that any losses due to default are first debited from the holdings of tranche B, the capital of tranche A being breached only when tranche B's capital is exhausted. That is

- If none of the bonds defaults, tranche A receives a coupon c_A plus the principle 1 and tranche B receives a coupon c_B plus the principle 1, where we must have $(c_A + 1) + (c_B + 1) = 2(c + 1)$.

- If both bonds default, tranche A receives $0(c_A + 1)$ and tranche B receives $0(c_B + 1)$, that is the recovery rate for both tranches is $R_1 := 0$.

- If exactly one of the bonds defaults, tranche A receives a coupon c_A plus the principle 1 and tranche B receives $R_2(c_B + 1)$, where we must have

$$(c_A + 1) + R_2(c_B + 1) = c + 1. \tag{1}$$

(Remember that if one of the constituent bonds defaults it has a recovery rate of 0.)

Tranche A is like a bond that pays coupon c_A and has a recovery rate $R_2 = 0$ in the case of default. But tranche B has three different payoffs depending on the three different possibilities for the two constituent bonds to default. We will look at tranche B again as a bond that pays coupon c_B but that has two different recovery rates, R_1 and R_2, depending on the two different ways for B to default.

Tranche A will experience a default event only if both underlying bonds default, which will happen with probability p^2. In this case, the recovery rate will be $R_1 = 0$. So, if tranche A is to pay the minimum palatable coupon rate to attract expected value investors, it must pay

$$c_A := \frac{r + p^2}{1 - p^2}.$$

Exercise 1. *Show that $c_A \leq c$ and equality holds if and only if $p = 0$ or 1.*

The statement of the exercise makes intuitive sense: the coupon is smaller for tranche A since the risk of default is smaller $p^2 \leq p$ (with equality if and only if $p = 0$ or 1).

If none of the bonds defaults, then we have \$$2c$ dollars to distribute and the coupon remaining to pay tranche B is determined by the equation $c_A + c_B = 2c$. In that case, it is easy to see that

$$c_B := \frac{2p(1+r) + r + p^2}{1 - p^2}.$$

(Note that $c_B \geq c_A$.) If both bonds default, the probability of this occurring is p^2, and then the recovery rates, for both tranches is $R_1 = 0$ (this was the case that also wiped out tranche A).

What remains for us to find is the recovery rate R_2. We do that by considering the third remaining case when exactly one of the two underlying bonds defaults. That happens with probability $2p(1-p)$. We can see right away that $R_2 \neq 0$ since the coupon, c_A, paid to the unaffected tranche A is less than the coupon, c, paid by the bond which did not default. That is, the bond that did not default pays us c and we pay c_A to the holder of tranche A. The difference $c - c_A$ is paid to tranche B. Using equation (1), we find that

$$\begin{aligned} R_2 = \frac{c - c_A}{1 + c_B} &= \frac{\left(\frac{r+p}{1-p}\right) - \left(\frac{r+p^2}{1-p^2}\right)}{1 + \left(\frac{2p(1+r)+r+p^2}{1-p^2}\right)} = \frac{p(1+r)}{(1-p^2) + 2p(1+r) + r + p^2} \\ &= \frac{p(1+r)}{1 + r + 2p(1+r)} = \frac{p(1+r)}{(1+r)(1+2p)} = \frac{p}{1+2p}. \end{aligned}$$

3 The multiperiod case

3.1 Setup of the problem

Suppose now we have two N-period bonds with principle X (that is we paid \$$X$ to buy the bonds) that pay coupons

$$C := cX$$

at times 1 through $N-1$. At time N, if there is no default the lender pays back $X + cX$. The probability of default at each period is p and the periods, as well as the two bonds, are independent from each other. Suppose the recovery rate of the bonds is $R = 0$. We issue a collateralized debt obligation consisting of two tranches: senior Tranche A and junior Tranche B. We sell the tranches for \$$X$ each and that covers what we paid for the bonds. Hence, every period, the coupons from the underlying two bonds have to be distributed to the tranches. We want to make tranche A less risky and we pay its coupon first, while tranche B is more risky and pay its holder what ever is left.

• If none of the underlying bonds defaults then each tranche gets its coupon c_A and c_B, which must satisfy $c_A + c_B = 2c$.

• If both underlying bonds default then both tranches default as well and that is the end of their lives. The recovery rate R_1 of the tranches should satisfy $R_1(c_A+1)+R_1(c_B+1) = 0\times 2(c+1)$. That is, $R_1 = 0$. This is default of type 1 and it happens with probability p^2.

• If one of the underlying bonds defaults then tranche B defaults with recovery rate R_2 and that is the end of its life. The recovery rate of the tranches should satisfy $c_A + R_2(c_B + 1) = c$. Tranche A continues to exist until the remaining bond pays its coupons. But now we have to pass the coupon c of the remaining bond entirely to tranche A. When, and if, that bond defaults, so does tranche A and that is the end of its life. This is default of type 2 and it happens with probability $2p(1-p)$. After that default the coupon of tranche A, as we will see, increases from c_A to c, but so does the probability of default of tranche A, from p^2 to p. As we will see the increased coupon compensates exactly for the increased risk and the formula for the coupon c_A will be exactly the same as the one in the single period model. Note that if the increased coupon c is not enough to compensate for the increased risk, them we may have to bump up the size of the coupon c_A that tranche A was receiving before. So it is not a priori clear that the formula for c_A, and hence c_B, should be the same as in the single period model.

So, we have two equations

$$c_A + c_B = 2c \quad \text{and} \quad (c_A + 1) + R_2(c_B + 1) = c + 1 \tag{2}$$

with three unknowns c_A, c_B, and R_2. Seems an impossible task to solve. We will derive a third equation relating c_B and R_2 using the fact that tranche B is like a bond but this time with two different default types and recovery rates. Finding the coupon c_B of that bond so that it is prized at par will give the desired relationship.

3.2 Valuation of a bond with two different default types

We begin by referring to our difference equation for valuing risky bonds. Assume that three mutually exclusive events can occur. We have a bond with principle X that pays coupons

$$C := cX$$

at times 1 through $N-1$. At time N, if there is no default the lender pays back $X + cX$. Assume now that the bond can default in two ways.

Default type 1 may occur in any period with probability p_1 and has associated recovery rate R_1. (That is, at the end of the period the lender pays back $R_1(X + C)$.)

Default type 2 may occur in any period with probability p_2 and has associated recovery rate R_2. (That is, at the end of the period the lender pays back $R_2(X + C)$.) No default occurs with probability $1 - p_1 - p_2$.

• Let V_0 be the expected value of the bond immediately after the principle X is lent.

• Let V_k be the expected value of the bond immediately after the k-th payment is made, $k = 1, \ldots, N-1$. That is, there are $N-k$ remaining payments. Note that V_k contains all the information about possible defaults at times $k+1$, $k+2, \ldots, N$.

• Assume that the principal is repaid a tiny bit later than the final coupon, so that V_N is the value of the principal, that is, let $V_N = X$.

A derivation analogous to the one used when we evaluated a bond with one type of default gives us the relationship

$$\begin{aligned} V_k &= (1-p_1-p_2)\frac{C+V_{k+1}}{1+r} + p_1\frac{R_1(C+X)}{1+r} + p_2\frac{R_2(C+X)}{1+r} \\ &= (1-p_1-p_2)\frac{C+V_{k+1}}{1+r} + (p_1+p_2)\Big(\frac{p_1R_1+p_2R_2}{p_1+p_2}\Big)\Big(\frac{C+X}{1+r}\Big) \end{aligned} \tag{3}$$

Define a new probability

$$P := p_1 + p_2$$

and a new recovery rate

$$S := \frac{p_1R_1+p_2R_2}{p_1+p_2}$$

and observe that the recursive relationship becomes

$$V_k = (1-P)\frac{C+V_{k+1}}{1+r} + P\frac{S(C+X)}{1+r}$$

with $V_N = X$. This is the same relationship that we solved when we valued an ordinary bond having probability of default P and a recovery rate of S. It is therefore true that the coupon rate which must be paid on a par bond to make it minimally palatable to a risk-neutral investor will be

$$c = \frac{r+P(1-S)}{1-P(1-S)}. \tag{4}$$

We may use this equation to determine the "fair" coupon, c_B, of tranche B in order for it to be sold at par.

Exercise 2. *Find the coupon of a bond with two different default types in any period. Default type* 1 *occurs with probability* p_1 *and has recovery rate* R_1*. Default type* 2 *occurs with probability* p_2 *and has recovery rate* 1*. No default occurs with probability* $1-p_1-p_2$*.*

Does the answer remind you of something? Can you explain the coincidence intuitively.

Exercise 3. *Find the at-par coupon of a bond with three different default times occurring with probabilities* p_1, p_2, p_3 *and having corresponding recovery rates* R_1, R_2, R_3*. No default occurs with probability* $1-p_1-p_2-p_3$*.*

Now apply this general formula to the collateralized debt obligation setting. Tranche B is a bond with two different defaults. Its coupon is c_B and the two recovery rates are $R_1 = 0$ and R_2, to be determined. Moreover, the probabilities of default are $p_1 = p^2$ and $p_2 = 2p(1-p)$. Then,

$$P = p_1 + p_2 = 2p - p^2,$$

$$S = \frac{p_1 R_1 + p_2 R_2}{p_1 + p_2} = \frac{2p(1-p)R_2}{2p - p^2} = \frac{2(1-p)R_2}{2-p}.$$

Hence

$$c_B = \frac{r + P(1-S)}{1 - P(1-S)} = \cdots = -\frac{(2p - 2p^2)R_2 - r - 2p + p^2}{(2p - 2p^2)R_2 + p^2 - 2p + 1},$$

where the dots indicate that simplifications are taking place, after you plug in the expressions for S and P.

Exercise 4. *Fill in the dots above.*

So

$$c_B = -\frac{(2p - 2p^2)R_2 - r - 2p + p^2}{(2p - 2p^2)R_2 + p^2 - 2p + 1} \tag{5}$$

is the third equation, that we need to add to (2) to obtain a system of three equations with three unknowns c_A, c_B, and R_2.

3.3 Finding the value of R_2

We need to solve the system of three equations

$$\begin{aligned} c_A + c_B &= 2c \\ (c_A + 1) + R_2(c_B + 1) &= c + 1 \\ c_B &= -\frac{(2p - 2p^2)R_2 - r - 2p + p^2}{(2p - 2p^2)R_2 + p^2 - 2p + 1} \end{aligned}$$

for the unknowns c_A, c_B, and R_2. The problem is that the system is not linear but it is still easy to solve. It is easy to eliminate c_A by solving for it in the first equation and substituting it out in the second:

$$\begin{aligned} c + R_2(c_B + 1) &= c_B \\ c_B &= -\frac{(2p - 2p^2)R_2 - r - 2p + p^2}{(2p - 2p^2)R_2 + p^2 - 2p + 1} \end{aligned}$$

Solving the first equation for c_B we get

$$c_B = \frac{c + R_2}{1 - R_2}$$

and substituting into the second equation gives

$$\frac{c + R_2}{1 - R_2} = -\frac{(2p - 2p^2)R_2 - r - 2p + p^2}{(2p - 2p^2)R_2 + p^2 - 2p + 1}.$$

After the cross-multiply, we end up with a linear equation for R_2, since the quadratic term cancels.

$$(2cp^2 - 1 - 2cp + 2p^2 - 2p - r)R_2 + (2cp - c + 2p - cp^2 + r - p^2) = 0.$$

It will be easier to collect all terms that contain c together

$$(-p^2 + 2R_2p^2 + 2p - 2R_2p - 1)c + r + 2R_2p^2 - 2R_2p - R_2 + 2p - p^2 - R_2r = 0$$

Now, recall that c is the coupon of the constituent two bonds each valued at par and each having recovery rate 0. So, we have

$$c = \frac{r+p}{1-p}.$$

Substitute in the equation for R_2 to get after some simplifications

$$(2pr + 2p + 1 + r)R_2 - (p + pr) = 0.$$

The solution is

$$R_2 = \frac{p}{2p+1}. \tag{6}$$

That is, the second recovery rate is exactly the same as in the single period model. The next subsection, finds the coupons c_A and c_B.

3.4 Finding the values of c_A and c_B

Substituting (6) into (5), we obtain

$$\begin{aligned} c_B &= -\frac{(2p-2p^2)p/(2p+1) - r - 2p + p^2}{(2p-2p^2)p/(2p+1) + p^2 - 2p + 1} \\ &= -\frac{(2p-2p^2)p + (-r-2p+p^2)(2p+1)}{(2p-2p^2)p + (p^2-2p+1)(2p+1)} \\ &= \frac{p^2 + 2pr + 2p + r}{p^2 - 1}. \end{aligned}$$

We see that the formula for c_B is exactly the same as in the single period model. Hence, using $c_A = 2c - c_B$ we can see that the formula for c_A stays the same as well:

$$c_A = 2c - c_B = 2\frac{r+p}{1-p} - \frac{r + 2p(1+r) + p^2}{1-p^2} = \cdots = \frac{r+p^2}{1-p^2}.$$

Exercise 5. *Show that $c \leq c_B$ and equality holds if and only if $p = 0$. Compare with Exercise 1. Why does this happen intuitively?*

Recall what we said about the tranches in the case when one of the underlying bonds defaults: Tranche B defaults and seizes to exist, while Tranche A continues to exist until the remaining bond pays its coupons. But now we have to pass the coupon c of the remaining bond entirely to tranche

A. When, and if, that bond defaults, so does Tranche A and that is the end of its life. Equivalently, we may think that at the time of when the underlying bond defaults, Tranche A receives its coupon c_A and then becomes an asset that is just like the remaining bond. But we know that the value of that bond at any time is 1 so Tranche A may just request to get all its remaining payments at the time when one of the bonds defaults and then seize to exist. The situation is equivalent to what we had above. Spelled out with cases, it is

• If none of the underlying bonds defaults then each tranche gets its coupon c_A and c_B, which must satisfy $c_A + c_B = 2c$.

• If both underlying bonds default then both tranches default as well and that is the end of their lives. The recovery rate R_1 of the tranches should satisfy $R_1(c_A+1)+R_1(c_B+1) = 0\times 2(c+1)$. That is, $R_1 = 0$. This is default of type 1 and it happens with probability p^2.

• If one of the underlying bonds defaults then tranche B defaults with recovery rate R_2 and that is the end of its life. Tranche A receives a coupon c_A and the present expected value of the of the future coupons and principle of the remaining bond, that is 1. The recovery rate of the tranches should satisfy $(c_A + 1) + R_2(c_B + 1) = c + 1$.

It is now immediately clear (Is it? Use one of the exercise above.) that Tranche A is a bond with coupon c_A, probability of default p^2, and recovery rate 0. Hence,

$$c_A = \frac{r + p^2}{1 - p^2}.$$

Tranche B is a bond with two different default types. Its coupon c_B can be calculated either using the method in Subsection 3.2 of using the equation $c_A+c_B = 2c$, where we know that $c = (r+p)/(1-p)$. Then equation $(c_A + 1) + R_2(c_B + 1) = c + 1$ gives us the recovery rate R_2.

Exercise 6. *Solve the system of three equations*

$$\begin{aligned} c_A + c_B &= 2c \\ (c_A + 1) + R_2(c_B + 1) &= c + 1 \\ c_A &= \frac{r + p^2}{1 - p^2} \end{aligned}$$

for the unknowns c_A, c_B, and R_2. Is the solution different that before?

Exercise 7. *Find the coupons and the recovery rates of Tranche A and Tranche B if the two underlying bonds have recovery rate $R > 0$. Assume that in the case of two default the recovery rate of both tranches is also R.*

4 Correlation of two identical bonds

In the final section of this chapter, we will investigate the impact of correlations on two bond CDOs.

In the earlier sections of this chapter we learned that collateralized debt obligations work very well at reducing risk in the senior tranche if the two constituent bonds have uncorrelated defaults. With this assumption, it is quite unlikely for both bonds to default, giving quite a lot of protection to the senior structure.

In this section we extend that work to the more realistic case in which defaults are correlated. Defaults tend to occur in response to the economic cycle: when times are tough, many companies may default on their debt, while during good times even relatively poorly managed companies can manage to survive.

Intuitively, introducing correlation between the defaults will reduce the protection to the senior tranche. In this section, we investigate the impact of correlation between defaults in a quantitative way.

Let us begin with some notation here:

- Let P_{ss} = probability bond 1 and bond 2 both survive;
- Let P_{sd} = probability bond 1 survives and bond 2 defaults;
- Let P_{ds} = probability bond 1 defaults and bond 2 survives;
- Let P_{dd} = probability bond 1 and bond 2 both default.

Clearly, we must have

$$P_{ss} + P_{sd} + P_{ds} + P_{dd} = 1.$$

In addition, each probability must be nonnegative. We assume that the two bonds have identical unconditional default probability, which means that

$$P_{ds} + P_{dd} = p = P_{sd} + P_{dd}.$$

So far we have three equations in four unknowns

$$\begin{aligned} P_{ss} + P_{sd} + P_{ds} + P_{dd} &= 1 \\ P_{ds} + P_{dd} &= p \\ P_{sd} + P_{dd} &= p \end{aligned} \tag{7}$$

together with the conditions

$$P_{ss},\ P_{sd},\ P_{ds},\ P_{dd} \geq 0.$$

Any solution (7) will depend on a single parameter (in addition to p). Let us see what is the correlation between the defaults of the two bonds and use the correlation as an additional parameter in order to express the joint probabilities $\{P_{ss}, P_{sd}, P_{ds}, P_{dd}\}$.

Define two random variables B_1 and B_2, where B_i is one if the bond i survives and zero otherwise, for $i = 1, 2$. The expected values are as follows:

$$E[B_1] = E[B_2] = 1 - p$$

and

$$Var[B_1] = Var[B_2] = (1-p)(1-(1-p))^2 + p(0-(1-p))^2 = (1-p)p^2 + p(1-p)^2 = p(1-p).$$

Now we find the correlation, ρ_{B_1,B_2}, between B_1 and B_2:

$$\rho_{B_1,B_2} = \frac{Cov[B_1, B_2]}{\sqrt{Var[B_1]Var[B_2]}} = \frac{Cov[B_1, B_2]}{p(1-p)}.$$

Next,

$$Cov[B_1, B_2] = E[(B1 - E[B_1])(B_2 - E[B_2])] = E[B_1 B_2] - E[B_1]E[B_2]$$
$$= (P_{ss} \cdot 1 + P_{ds} \cdot 0 + P_{sd} \cdot 0 + P_{dd} \cdot 0) - (1-p)^2$$

Substituting into the formula for ρ_{B_1,B_2} we get

$$\rho_{B_1,B_2} = \frac{P_{ss} - (1-p)^2}{p(1-p)}.$$

Let our new parameter be

$$\rho := \frac{P_{ss} - (1-p)^2}{p(1-p)}$$

or equivalently

$$P_{ss} = \rho p(1-p) + (1-p)^2 = (1-p)(\rho p + (1-p)) = (1-p)^2\Big(1 + \rho\frac{p}{1-p}\Big).$$

Since ρ is the correlation between two random variables, we have

$$-1 \le \rho \le 1.$$

Now, we find, P_{ds}, P_{sd}, P_{dd} using (7). Subtracting the second and the third equation, we obtain that $P_{sd} = P_{ds}$. The system reduces to

$$P_{ss} + 2P_{ds} + P_{dd} = 1$$
$$P_{ds} + P_{dd} = p$$

or substituting P_{ss}, we get

$$2P_{ds} + P_{dd} = 1 - (1-p)(\rho p + (1-p))$$
$$P_{ds} + P_{dd} = p$$

Subtracting the second equation from the first, we get

$$P_{ds} = (1-p)(1 - \rho p - (1-p)) = p(1-p)(1-\rho).$$

Finally,

$$P_{dd} = p - P_{ds} = p - p(1-p)(1-\rho) = p(1-(1-p)(1-\rho)) = p(p\rho - p - \rho) = p^2\Big(1 + \rho\frac{1-p}{p}\Big).$$

To summarize, we found that the joint probabilities of default must satisfy

$$P_{ss} = (1-p)^2\Big(1 + \rho\frac{p}{1-p}\Big), \;\; P_{sd} = P_{ds} = p(1-p)(1-\rho), \;\; P_{dd} = p^2\Big(1 + \rho\frac{1-p}{p}\Big),$$

where ρ is the correlation between the two bonds. But, recall that probabilities need to be non-negative numbers. This puts restrictions on the possible values of ρ. Clearly, $P_{sd} \geq 0$ and $P_{ds} \geq 0$, since ρ is a correlations. But in order to have $P_{dd} \geq 0$ and $P_{ss} \geq 0$ we need to have

$$\rho \geq -\frac{p}{1-p} \text{ and } \rho \geq -\frac{1-p}{p}. \tag{8}$$

Let us see now how much of restrictions these really are. First, note that

$$-\frac{p}{1-p} \leq 1 \text{ and } -\frac{1-p}{p} \leq 1$$

as well as

$$-\frac{p}{1-p} \geq -\frac{1-p}{p} \text{ if and only if } p \leq 1/2$$

with equality holding if and only if $p = 1/2$. So we consider two cases.

Case 1. If $p \leq 1/2$, then conditions (8) are satisfied precisely when

$$\rho \geq -\frac{p}{1-p}$$

is the only condition we need to impose on ρ. (Note that in this case $-\frac{p}{1-p} \geq -1$.)

Case 2. If $p \geq 1/2$, then conditions (8) are satisfied precisely when

$$\rho \geq -\frac{1-p}{p}$$

is the only condition we need to impose on ρ. (Note that in this case $-\frac{1-p}{p} \geq -1$.)

Finally, we compare with the joint probabilities of default when the two bonds are uncorrelated. That is, if the two bonds default independently define

$$P^\circ_{ss} = (1-p)^2, \;\; P^\circ_{ds} = p(1-p), \;\; P^\circ_{sd} = (1-p)p, \;\; P^\circ_{dd} = p^2. \tag{9}$$

Notice that

$$\frac{P_{ss}}{P^\circ_{ss}} = 1 + \rho\frac{p}{1-p}, \;\; \frac{P_{ds}}{P^\circ_{ds}} = 1 - \rho, \;\; \frac{P_{dd}}{P^\circ_{dd}} = 1 + \rho\frac{1-p}{p}.$$

4.0.1 The coupons of the tranches in the correlated case

With these probabilistic preliminaries behind us, we can work out the payouts of a CDO tranche. For simplicity, assume that for both bonds $R = 0$ and the senior tranche supplies half the capital to the structure. Then the senior tranche pays off in full if at least one of the bonds survives. This means that the senior tranche has default probability

$$p^2\Big(1 + \rho\frac{1-p}{p}\Big) = p^2 + \rho p(1-p).$$

With this default probability the fair coupon for the senior tranche should be

$$c_A := \frac{r + p^2 + \rho p(1-p)}{1 - p^2 - \rho p(1-p)}, \tag{10}$$

which if $\rho = 0$, reduces to the familiar

$$c_A := \frac{r + p^2}{1 - p^2}$$

but if $\rho = 1$ it reduces to the fair rate of the constituent bonds

$$c_A := \frac{r + p}{1 - p}.$$

What if ρ is as small as possible? Assuming $p < 1/2$, this means $\rho = -p/(1-p)$. Inserting that into equation (10) yields a fair senior tranche coupon of:

$$c_A := \frac{r + p^2 - p^2}{1 - p^2 + p^2} = r.$$

That is, we get a fair coupon of r since it is impossible to have two defaults:

$$P_{dd} = p^2\Big(1 + \rho\frac{1-p}{p}\Big) = 0.$$

As before, the coupon to the junior tranche can be obtained by subtracting the coupon payable to the senior tranche from the total coupons paid. Thus,

$$c_B = 2c - c_A = 2\frac{r+p}{1-p} - \frac{r + p^2 + \rho p(1-p)}{1 - p^2 - \rho p(1-p)}.$$

It turns out to be not particularly insightful to expand the last difference. But we can look at it numerically.

Exercise 8. *Analyze the situation when $p > 1/2$ and ρ is as small as possible.*

Exercise 9. *Find the coupons of the tranches when the recovery rate of the underlying bonds is $R > 0$. Assume that in the case of two default the recovery rate of both tranches is also R.*

Example 10. *Take $r = 3\%$, $p = 10\%$, $R = 0\$$. Then $p/(1-p) = 0.1/0.9 = 1/9$. The three coupons are: $c = (r+p)/(1-p) = 0.13/0.9 = 14.44\%$. Consider the $\rho = 0$ case first:*

$$c_{A,\rho=0} = \frac{3\% + 1\%}{1 - 1\%} = \frac{4\%}{0.99} = 4.04\%$$
$$c_{B,\rho=0} = 2(14.44\%) - 4.04\% = 24.84\%.$$

This is really the case at the beginning of the chapter. Here, the senior tranche is very safe, and defaults only with probability 1%, hence pays just about 1% more than the risk-free rate. But the junior tranche is very dangerous, as it absorbs defaults about 20% of the time, hence its large coupon.

Consider now the $\rho = 1$ case:

$$c_{A,\rho=1} = \frac{3\% + 10\%}{1 - 10\%} = \frac{13\%}{0.9} = 14.44\%$$
$$c_{B,\rho=1} = 2(14.44\%) - 14.4\% = 14.44\%.$$

In the case of perfect correlation between bonds, the tranching inherent in the CDO does not really do anything. (There are not even any diversification benefits to be had.) So the two tranches have identical properties. This result may at first seem a bit counterintuitive, in that it suggests that correlation reduces the risk (and hence the return) for the junior tranche. This is in fact true in this framework, where the total overall number of defaults to be expected is the same with or without correlation. The impact of the correlation is to make the senior. tranche pay for more of these defaults, which means that the junior tranche pays for fewer.

Finally, let $\rho = -11.1\%$*:*

$$c_{A,\rho=-1/9} = 3\%$$
$$c_{B,\rho=-1/9} = 2(14.44\%) - 3\% = 25.88\%.$$

Here the senior tranche is totally bulletproof, since there is no way that both bonds can default at the same time. As such, it pays just the risk-free rate. Now ALL the costs of default are absorbed by the junior tranche, to a degree even a bit more than in the uncorrelated case. This makes the junior tranche a bit riskier, and hence gives it more return.