FM 3613 — Mathematics of Financial Options

Chapter 7 - Bond Pricing with Default: Using Difference Equations

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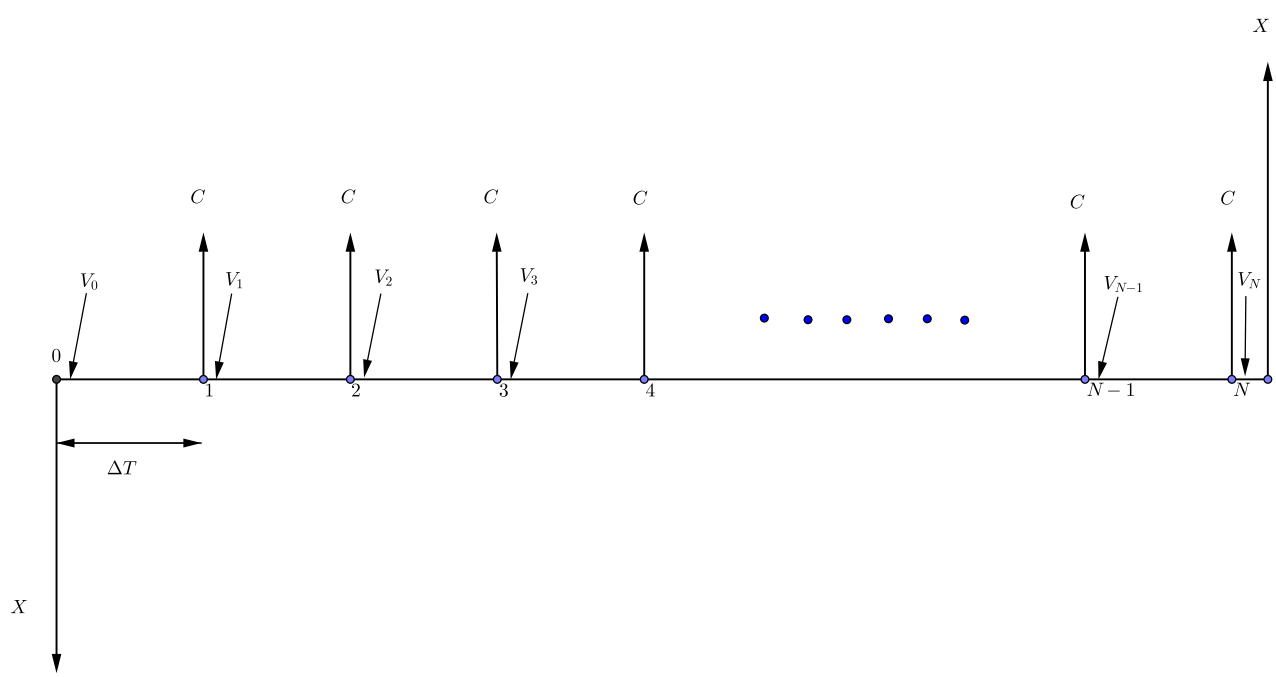
1 Using difference equations to find the coupon, C, of a bond

• Let V_0 be the expected value of the bond immediately after the principle X is lent.

• Let V_k be the expected value of the bond immediately after the k-th coupon payment is made, $k = 1, \ldots, N-1$. That is, there are $N-k$ remaining payments. Note that V_k contains all the information about possible defaults at times $k+1$, $k+2, \ldots, N$.

• Assume that the principal is repaid a tiny bit later than the final coupon, so that V_N is the value of the principal, that is, let $V_N = X$.



This separation of the final coupon and the principal repayment is the first little trick of this problem. The second trick is to set up the problem backwards in time.

In order to compute V_k, all we need to consider is the possibility of a default at time $k+1$ as the effect of all the other default events, after that, are lumped into the value V_{k+1}. A default at time $k+1$ occurs with probability p, returning $R(C+X)$ at time $k+1$. Here R is a fixed number in $[0, 1]$.

If there is no default (with probability $1-p$), a coupon of C is received at time $k+1$ and the remaining payments, adjusted for the possibility of future default, are worth V_{k+1}.

Note that the possibility of default does not affect the value of V_N since it is not possible to have a default at time $N+1$.

Since, all of these payments are at time $k+1$, we need to bring them back to time k by dividing them by $1+r$ to account for the time value of money.

Since the lender simply computes the expected value of her cash flows, we can write down the difference equation:

$$\begin{aligned} V_k &= (1-p)\frac{C+V_{k+1}}{1+r} + p\frac{R(C+X)}{1+r} \quad \text{for } k = 0, \ldots, N-1, \\ V_N &= X. \end{aligned}$$

Our objective is to find the value of the bond at issue, V_0.

To solve the problem, we non-dimensionalize by letting $w_k := V_k/X$ and $c := C/X$, where c now has the interpretation of a "coupon rate."

$$\begin{aligned} w_k &= (1-p)\frac{c+w_{k+1}}{1+r} + p\frac{R(c+1)}{1+r} \quad \text{for } k = 0, \ldots, N-1, \\ w_N &= 1. \end{aligned}$$

We also make the time-reversal transformation, that is, we let $v_{N-k} := w_k$:

$$v_{N-k} = (1-p)\frac{c+v_{N-k-1}}{1+r} + p\frac{R(c+1)}{1+r} \quad \text{for } N-k = 1, \ldots, N,$$

and note that the initial condition becomes $v_0 = w_N = 1$. Finally, let $j := N-k-1$ and note that $j = 0, \ldots, N-1$ as $N-k = 1, \ldots, N$. We have

$$\begin{aligned} v_{j+1} &= (1-p)\frac{c+v_j}{1+r} + p\frac{R(c+1)}{1+r} \\ &= \frac{c(1-p(1-R))}{1+r} + \frac{pR}{1+r} + \frac{1-p}{1+r}v_j. \end{aligned}$$

Our goal is to find v_N.

In order to simplify the notation, define

$$a := \frac{1-p}{1+r} \text{ and } b := \frac{c(1-p(1-R))}{1+r} + \frac{pR}{1+r}$$

and then we simply have

$$v_{j+1} = av_j + b$$

$$v_0 = 1.$$

We already know how to solve such and equation and the solution is

$$v_j = a^j + \frac{1-a^j}{1-a} b \text{ for all } j = 0, \ldots, N.$$

Thus, our goal to find v_N is achieved

$$v_N = a^N + \frac{1-a^N}{1-a} b,$$

from where we get

$$V_0 := X\Big(a^N + \frac{1-a^N}{1-a} b\Big).$$

When a and b are substituted back in, this becomes a fairly complicated expression.

Now we ask another question: What should the coupon rate be for the bond to be worth "par" at issue. This means, what should C be in order that $V_0 = X$. That is, when we lend the principle amount X, the bond that we get in return should also be valued at X, that is only fair.

In our time-reversed non-dimensional coordinates, the condition $V_0 = X$ is equivalent to $v_N = 1$ or

$$a^N + \frac{1-a^N}{1-a} b = 1.$$

Equivalently

$$\frac{1-a^N}{1-a} b = 1 - a^N$$

or after cancelling $1 - a^N$, we are left with

$$b = 1 - a.$$

Now recall the definitions of a and b, to obtain

$$1 - \frac{1-p}{1+r} = \frac{c(1-p(1-R))}{1+r} + \frac{pR}{1+r}$$

or

$$(1+r) - (1-p) = c(1-p(1-R)) + pR.$$

Solving for c, we get

$$c = \frac{r + p(1-R)}{1-p(1-R)}.$$

2 Insights arising from the formula for c

Case 1. If the probability of default p is zero, then we get $c = r$. This makes perfect sense — if defaults are impossible there is no reason to pay above the risk-free rate.

Case 2. Note that the formula for c does not depend on N. That means that as long as the probability of default p, the recovery rate R, and the interest r over time period ΔT, stay the same, the bond may have any number of repayments and the size of the coupon will stay the same.

Case 3. If the recovery rate, R, is equal to 1, then $c = r$ as well. This also makes sense — a "default" in which the company makes good on all its debts is not really a default but a prepayment. Default in that case means that the bond's life is shortened. But c does not depend on N.

Now for real life bonds the ability of a debtor to prepay her debts does require an increased coupon rate, because in the real world interest rates fluctuate and this prepayment feature gives its holder the ability to refinance debt when interest rates fall.

- The term $p(1-R)$ is called the **risk metric** for this problem.

- The **yield spread** is defined to be $c - r$:

$$c - r = \frac{r + p(1-R)}{1 - p(1-R)} - r = \frac{p(1-R)(1+r)}{1 - p(1-R)} > p(1-R).$$

Exercise 1. *Suppose you lend \$X dollars to buy a bond with N coupons C, priced at par, and probability of default p in each period. Suppose that just after the k-th coupon you decide to cell your bond. How much should you fairly ask for if the interest rate stays the same?*

Stop and think about the last exercise. It is extremely important.

Exercise 2. *Suppose you lend \$X dollars to buy a bond with N coupons C, priced at par, and probability of default p in each period. Suppose that just after the k-th coupon the probability of default per period goes up. (Your remaining coupons will stay at C dollars per period.) What will happen to the value of your bond? What will happen if the probability of default goes down?*

Exercise 3. *Suppose you lend \$X dollars to buy a bond with N coupons C, priced at par, and probability of default p in each period. Suppose that just after the k-th coupon the interest rate per period goes up. (Your remaining coupons will stay at C dollars per period.) What will happen to the value of your bond? What will happen if the interest rate goes down?*