1 Basic definitions in linear programming

The following problem will be our standing testing example.

The Strawberry Problem: A factory manufactures two products, strawberry jam and strawberry drink. It requires three inputs: fresh strawberries, water, and electricity.

Each unit of strawberry jam earns a profit of \$2. To make it requires one unit of fresh strawberries and two units of electricity, and one unit of water appears as a byproduct. The water that appears as a by product may be used in the production of strawberry juice.

Each unit of strawberry juice earns a profit of \$3. To make it requires one unit of fresh strawberries, one unit of water and one unit of electricity.

Each day the factory has available six units of strawberries and four units of water and ten units of electricity.

How much strawberry juice and strawberry jam should the factory make each day in order to maximize profit?

Formulation of the linear model: Let x_1 be the number of units of strawberry jam made each day. Let x_2 be the number of units of strawberry juice made each day. The linear model is

$$(1) \qquad \begin{cases} \max & 2x_1 & + & 3x_2 & & \\ \text{subject to} & x_1 & + & x_2 & \leq & 6 \\ & -x_1 & + & x_2 & \leq & 4 \\ & 2x_1 & + & x_2 & \leq & 10 \\ & x_1 & , & x_2 & \geq & 0. \end{cases}$$

Let us show that $(1, 5)$ is an optimal solution. Suppose that (x_1, x_2) is any point that satisfies the constraints. Notice that the objective function can be bounded above by

$$2x_1 + 3x_2 = \frac{5}{2}(x_1 + x_2) + \frac{1}{2}(-x_1 + x_2) \leq \frac{5}{2}6 + \frac{1}{2}4 = 17.$$

This shows that the objective value of (x_1, x_2), satisfying the constraints, can never be larger than 17. Since $(1, 5)$ has objective value 17 it must be an optimal solution.

Of course, in order to apply this method we need to have a good guess about what the optimal solution might be, and even then, we need to express the objective function in terms of the left-hand sides of the constraints, which may be tricky. Another way to solver this problem is the graphical method that we are going to discuss shortly.

In general a linear problem may include

- minimization or maximization of a linear function: $z = c_1x_1 + c_2x_2 + \cdots + c_nx_n + c$;
- several linear constraints of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \left\{ \begin{array}{c} \leq \\ = \\ \geq \end{array} \right\} b;$$

- there may be upper and/or lower bounds on some or all of the variables, such as $l_i \le x_i \le L_i$, where we allow l_i to be $-\infty$ in which case there is no lower bound on x_i and we allow L_i to be $+\infty$ in which case there is not upper bound on x_i. If there is neither upper nor lower bound on x_i then we say that x_i is a *free* variable.

An LP problem is in *standard inequality form* if it is in the form

$$
\begin{array}{lrcrcccrcl}
\text{maximize} & z = c_1x_1 & + & c_2x_2 & + & \cdots & + & c_nx_n & & \\
\text{subject to} & a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & \le & b_1 \\
 & a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & \le & b_2 \\
 & \vdots & & & & & & \vdots & & \vdots \\
 & a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & \le & b_m \\
 & x_1 & , & x_2 & , & \dots & , & x_n & \ge & 0.
\end{array}
$$

The variables $x_1, ..., x_n$ are called the *decision variables*. The function z is called the *objective function*. The next m inequalities are the *functional constraints*. Finally, the last line are the *positivity constraints*, n of them. The numbers n and m are not related.

There are four important characteristics of the standard inequality form. 1) It is a maximization problem; 2) the objective function has no constant term; 3) all functional constraints are less-than-or-equal inequalities; 4) all decision variables are required to be non-negative.

An LP problem is in *standard equality form* if it is in the form

$$
\begin{array}{lrcrcccrcl}
\text{maximize} & z = c_1x_1 & + & c_2x_2 & + & \cdots & + & c_nx_n & & \\
\text{subject to} & a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
 & a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
 & \vdots & & & & & & \vdots & & \vdots \\
 & a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \\
 & x_1 & , & x_2 & , & \dots & , & x_n & \ge & 0.
\end{array}
$$

Here all functional constraints are equalities.

Throughout the course we will use the following notation

$$
c := (c_1, ..., c_n), \qquad x := \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}
$$

$$
A := \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}, b := \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}.
$$

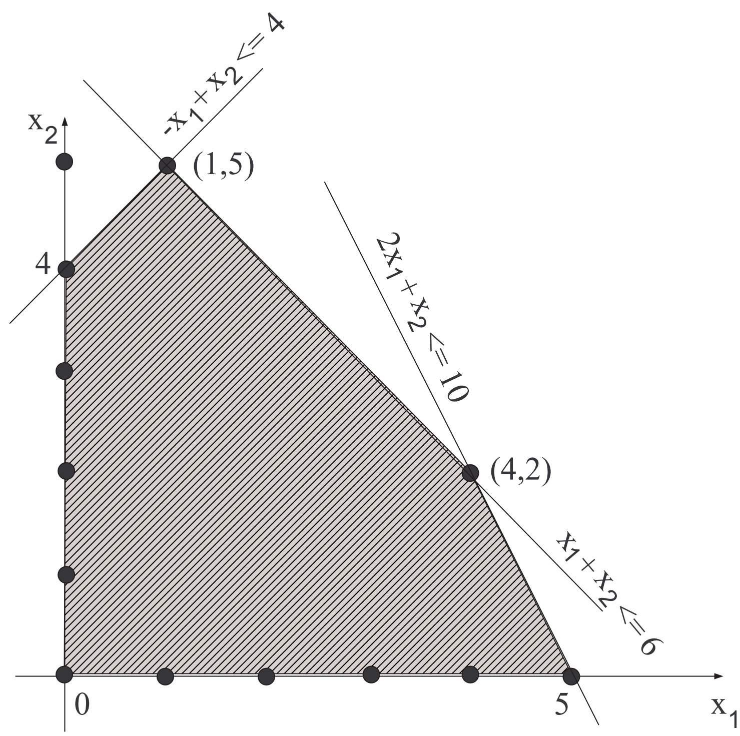
Clearly $c, x \in \mathbb{R}^n$ while $b \in \mathbb{R}^m$. With this notation the two standard forms can be written succinctly as

$$\begin{array}{lrcl} \max & z = cx & & \\ \text{s.t.} & Ax & \leq & b \\ & x & \geq & 0 \end{array} \qquad \text{and} \qquad \begin{array}{lrcl} \max & z = cx & & \\ \text{s.t.} & Ax & = & b \\ & x & \geq & 0. \end{array}$$

- Any $x \in \mathbb{R}^n$ is called a *solution* of the LP.
- If $x \in \mathbb{R}^n$ satisfies all of the constraints then it is called a *feasible solution* of the LP.
- If $x \in \mathbb{R}^n$ violates at least one of the constraints then it is called an *infeasible solution* of the LP.
- The set of all feasible solutions is called the *feasible region* of the LP.
- An $x \in \mathbb{R}^n$ is an optimal solution of an LP problem if it is a feasible solution and it yields the best objective value among all other feasible solutions. The best objective value is the largest one if the problem is a maximization one, or the lowest if the problem is a minimization one.
- The LP problem is called *infeasible* if it doesn't have any feasible solution, that is, if its feasible region is an empty set.
- The feasible region of an LP problem is *bounded* if it can be enclosed in a ball with large enough radius centered at the origin of the coordinate system. Conversely, the feasible region is *unbounded* if for any ball centered at the origin, with no matter how large radius, there are feasible points that are outside of the ball.
- An LP problem is called *unbounded* if we can find a sequence of feasible points having objective values converging to $+\infty$ (if the LP problem is a maximization one) or to $-\infty$ (if the LP problem is a minimization one).
- An LP problem is called *bounded* if has an optimal solution.

For example, the feasible region of (1) is the shaded region illustrated below. Notice that the feasible region of an LP problem depends only on the constraints and not on the objective function. The following three optimization problems are not LP problems.

$$\left\{ \begin{array}{lrcrcl} \max & 2|x_1| & + & 3x_2 & & \\ \text{s.t.} & x_1 & + & x_2 & \leq & 6 \\ & -x_1 & + & x_2 & \leq & 4 \\ & 2x_1 & + & x_2 & \leq & 10 \\ & x_1 & , & x_2 & \geq & 0. \end{array} \right.$$



This is not an LP problem because the objective function is not linear.

(2)
$$\left\{\begin{array}{lrcrcl}
\max & 2x_1 & + & 3x_2 & & \\
\text{s.t.} & |x_1 & + & x_2| & \leq & 6 \\
 & -x_1 & + & x_2 & \leq & 4 \\
 & 2x_1 & + & x_2 & \leq & 10 \\
 & x_1 & , & x_2 & \geq & 0.
\end{array}\right.$$

Similarly, the left-hand side of the first constraint is not a linear function.

$$\left\{\begin{array}{lrcrcl}
\max & 2x_1 & + & 3x_2 & & \\
\text{s.t.} & x_1 & + & x_2 & \leq & 6 \\
 & -x_1 & + & x_2 & \leq & 4 \\
 & 2x_1 & + & x_2 & \leq & 10 \\
 & x_1 & , & x_2 & > & 0.
\end{array}\right.$$

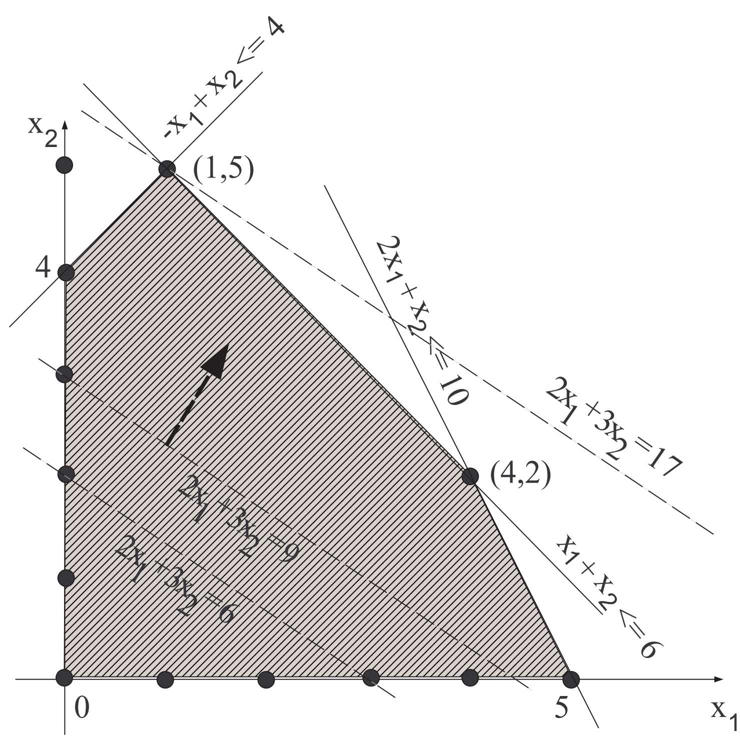
The third instance is not a linear problem because of the strict positivity requirements on the variables. Notice that optimization problem (2) can be converted into an equivalent LP problem:

$$\begin{cases} \max & 2x_1 & + & 3x_2 & & \\ \text{s.t.} & x_1 & + & x_2 & \leq & 6 \\ & x_1 & + & x_2 & \geq & -6 \\ & -x_1 & + & x_2 & \leq & 4 \\ & 2x_1 & + & x_2 & \leq & 10 \\ & x_1 & , & x_2 & \geq & 0. \end{cases}$$

2 The graphical method for solving a linear program

2.1 Unique solution

Since problem (1) involves only two variables we were able to plot its feasible region in the plane. We can use that graphical representation to completely solve the strawberry problem. The objective function $z = 2x_1 + 3x_2$ can be thought of as a family of parallel lines, one line for each fixed value of z. For example, on the figure below, the lines $6 = 2x_1 + 3x_2$ and $9 = 2x_1 + 3x_2$ are superimposed on the feasible region, represented as dashed lines. For any fixed value of z the line $z = 2x_1 + 3x_2$ is



called the *level line* of the objective function at level z. Looking at the figure, we notice that when z increases the level lines move in the direction indicated by the arrow. Since the LP problem is a maximization problem, our goal is to find the largest value of z for which the level line $z = 2x_1 + 3x_2$ intersects the feasible region. Clearly this is done by the level line $17 = 2x_1 + 3x_2$ and the intersection between this line and the feasible region contains only one point: $(1, 5)$. Thus, the graphical method shows that the strawberry problem has exactly one optimal solution with optimal value 17.

2.2 Infeasible LP

Let us consider the LP problem

$$\left\{\begin{array}{lrcrcr} \max & x_1 & + & x_2 & & \\ \text{s.t.} & -x_1 & + & x_2 & \leq & 1 \\ & x_1 & + & x_2 & \leq & -3 \\ & & & x_2 & \geq & 0. \end{array}\right.$$

It is easy to see algebraically that there is no point (x_1, x_2) that satisfies all the constraints. Indeed, if you add the two functional constraints we obtain $2x_2 \leq -2$ or equivalently $x_2 \leq -1$ which cannot be simultaneously satisfied with the positivity requirement $x_2 \geq 0$. Thus the LP problem is infeasible.

2.3 Unboundedness

Let us consider the LP problem

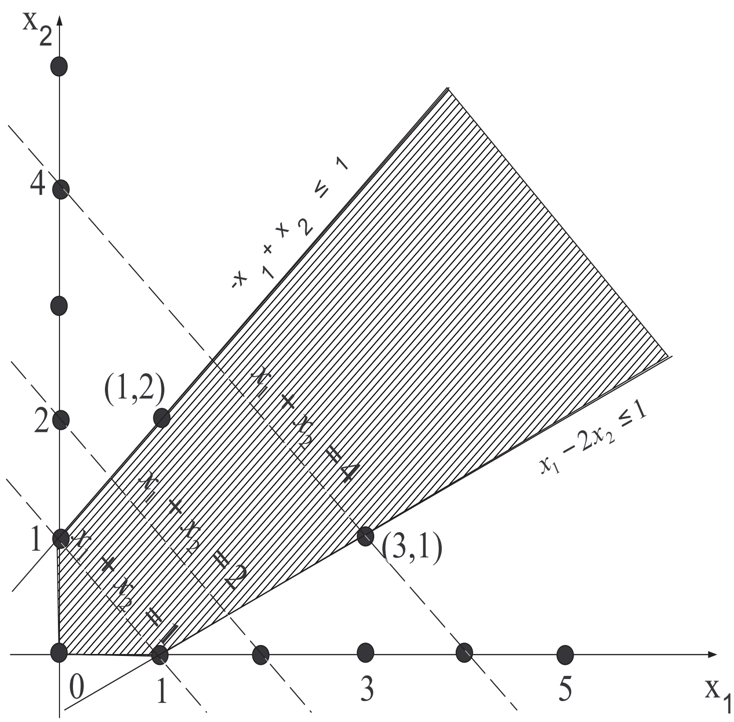
$$\left\{\begin{array}{lrcrcr} \max & x_1 & + & x_2 & & \\ \text{s.t.} & -x_1 & + & x_2 & \leq & 1 \\ & x_1 & - & 2x_2 & \leq & 1 \\ & x_1 & , & x_2 & \geq & 0. \end{array}\right.$$

The feasible region and the level lines are given on the next figure. It is easy to see, that no matter how large z is, the level line $z = x_1 + x_2$ will intersect the feasible region. That is, there are feasible points with objective values z. This shows that this LP problem is unbounded. Algebraically, let

$$x_1(t) := 1 + 2t, \;\; x_2(t) := t,$$

where t is a positive parameter. It is easy to see that the point $(x_1(t), x_2(t))$ is feasible for every $t \geq 0$. Its objective value is

$$z(t) = x_1(t) + x_2(t) = 1 + 3t \to +\infty \quad \text{as} \quad t \to +\infty.$$



2.4 Many optimal solutions

A slight change in the objective function of the last LP problem changes its properties drastically. It is easy to see that the problem

$$\begin{cases} \max & -x_1 & + & x_2 & & \\ \text{s.t.} & -x_1 & + & x_2 & \leq & 1 \\ & x_1 & - & 2x_2 & \leq & 1 \\ & x_1 & , & x_2 & \geq & 0 \end{cases}$$

has infinitely many optimal solutions. In fact every point on the ray $\{(0,1) + t(1,1) \,|\, t \geq 0\}$ is an optimal solution. (Why?)

It is important to see the following implications. If the feasible region of an LP problem is bounded then the LP problem has an optimal solution. If the LP problem is unbounded then its feasible region is unbounded. Whether an LP problem is unbounded or not depends on both the objective function and the constraints.

3 Converting any LP problem in standard forms

An LP problem has two standard forms: standard inequality form (SIF) and standard equality form (SEF). In this section we explain how to convert any LP problem into *any* of the two standard forms.

There are several operations that we can perform on an LP problem. Each operations changes the LP problem into a new problem equivalent to the old one. The old and the new problems are equivalent in the sense that if one of them is infeasible, or unbounded, or has an optimal solution then so is the other. Moreover, the transformations that we will apply are simple enough so that knowing an optimal solution of one of the LP problems easily gives us a solution of the other one.

(i) Transformations on the objective function.

- Adding or subtracting a constant to the objective function doesn't change the feasible region, nor the set of optimal solutions. It only changes the optimal value. Of course, the optimal value of the original problem can be recovered by subtracting or adding the same constant to the optimal value of the new problem.
- One can convert a minimization problem into a maximization one (and vice versa) by multiplying the objective function by (-1). The feasible region and the optimal solutions of the old and the new problems are the same. The optimal value of the original problem is minus the optimal value of the new one.

(ii) Transformations on the constraints.

- Every greater-than-or-equal constraint is equivalent to a less-than-or-equal constraint (and vice versa) obtained after multiplying the original one by (-1). This operation doesn't change anything about the LP problem except its appearance.
- Every equality constraint is equivalent to two inequality constraints. For example, the set of all vectors (x_1, x_2) satisfying the constraint $2x_1 - 3x_2 = 4$ is exactly the same as the set of all vectors (x_1, x_2) satisfying the pair of inequalities $2x_1 - 3x_2 \le 4$ and $-2x_1 + 3x_2 \le -4$.
- Every inequality constraint can be converted into an equality constraint with the help of one additional variable, called *the slack variable* for that inequality. The procedure is best described with an example. A vector (x_1, x_2, x_3) satisfies $x_1 + 2x_2 - 3x_3 \le 5$ if and only if the vector (x_1, x_2, x_3, s) satisfies $x_1 + 2x_2 - 3x_3 + s = 5$ and $s \ge 0$. Here, $s := 5 - x_1 - 2x_2 + 3x_3$ is the slack in the inequality constraint, hence the name of the new variable s. If we have more than one inequality constraints that we want to convert into equalities, we need to introduce one slack variable for each inequality constraint. This transformation changes the feasible region of the problem. The dimension of the feasible region of the new problems is one more than the dimension of the feasible region of the old problem. The connection between them is clear: (x_1, x_2, x_3, s) is feasible in the new problem with objective value v then (x_1, x_2, x_3) is feasible in the old problem with the same objective value v. (Note that we have not changed the objective function.) Conversely, if (x_1, x_2, x_3) is feasible in the old problem then $(x_1, x_2, x_3, 5 - x_1 - 2x_2 + 3x_3)$ is feasible in the new problem. If $(x_1^*, x_2^*, x_3^*, s^*)$ is an optimal solution of the new problem then (x_1^*, x_2^*, x_3^*) is an optimal solution of the old problem.

(iii) Transformations on the variables.

- If a variable is required to be non-positive, say $x_i \leq 0$ then we can replace every occurrence of x_i in the LP problem by $x'_i := -x_i$. Clearly, $x'_i \geq 0$.
- Recall that a variable is called *free* if it is not required to be non-negative in your original LP problem. Any such variable can be replaced by the difference of two non-negative new variables. In other words, if x_i is not required to be non-negative we replace every occurrence of it (both in the constraints and in the objective function) by $u_i - v_i$ where we require that $u_i \geq 0$ and $v_i \geq 0$. Thus if vector $(x_1, ..., x_i, ..., x_n)$ is a feasible solution of the original LP problem then $(x_1, ..., u_i, v_i, ..., x_n)$ is a feasible solution of the new problem with the same objective value, where

$$\begin{cases} u_i := x_i \text{ and } v_i := 0 & \text{if } x_i \geq 0 \\ u_i := 0 \text{ and } v_i := -x_i & \text{if } x_i < 0. \end{cases}$$

  Keep in mind that $x_i = u_i - v_i$. Conversely, if $(x_1, ..., u_i, v_i, ..., x_n)$ is a feasible solution of the new problem then $(x_1, ..., u_i - v_i, ..., x_n)$ is a feasible solution of the old problem with the same objective value.
- If you have an inequality constraint involving only one variable, say $x_i \leq L_i$ (here L_i is a constant) then you can *simplify* the problem by letting $x'_i := L_i - x_i$ and substitute every occurrence of x_i in the problem by $L_i - x'_i$. Clearly the constraint $x_i \leq L_i$ becomes a non-negative requirement for x'_i, that is $x'_i \geq 0$. If there is a constraint $l_i \leq x_i$ then we let $x'_i := x_i - l_i$ and substitute out x_i by $x'_i + l_i$. Don't forget to require $x'_i \geq 0$ again. If you have a double bound on a variable: $l_i \leq x_i \leq L_i$ then we can let $x'_i := x_i - l_i$ and replace every instance of x_i in the LP problem by $x'_i + l_i$. In particular, the double inequality $l_i \leq x_i \leq L_i$ becomes $0 \leq x'_i \leq L_i - l_i$ and we treat it as two constraints: the positivity constraint $0 \leq x'_i$ together with a functional constraint $x'_i \leq L_i - l_i$. Of course, you can treat the original double inequality $l_i \leq x_i \leq L_i$ as two functional constraints $-x_i \leq -l_i$ and $x_i \leq L_i$.

(iv) Free variables in an equality constraint. Suppose that x_i is a free variable and appears in an equality constraint of the LP problem. Then we can solve that constraint for x_i, thus expressing it in terms of the other variables, and then we can eliminate x_i from the problem by substituting it out of every other constraint including the objective function. Then we throw that equality constraint out of the problem. Well, we actually, put it aside, because it will be needed if you want to find out what the optimal value of x_i in the optimal solution should be. This transformation simplifies the problem because it reduces the number of variables by one.

Example 1. *Convert the following LP problem into standard inequality form.*

$$\begin{array}{lrcrcl} min & -2x_1 & + & 3x_2 & + & 7 \\ s.t. & x_1 & + & x_2 & = & 2 \\ & 4x_1 & + & 6x_2 & \geq & 9 \\ & x_1 & & & \geq & 0. \end{array}$$

First we remove the constant 7 from the objective function. Solving the problem with that constant or without it is one and the same thing. You can always add the value 7 to the optimal solution of the new problem. Then we multiply the objective function by (-1) and change min to max. The result is:

$$\begin{array}{lrcrcl} max & 2x_1 & - & 3x_2 & & \\ s.t. & x_1 & + & x_2 & = & 2 \\ & 4x_1 & + & 6x_2 & \geq & 9 \\ & x_1 & & & \geq & 0. \end{array}$$

After we solve this problem, its optimal value now has to be multiplied by (-1) and then we should add 7 to get the optimal value of the original one. The optimal solutions are the same.

Next, we replace the equality constraint by two inequalities with opposite signs:

$$\begin{array}{lrcrcl} max & 2x_1 & - & 3x_2 & & \\ s.t. & x_1 & + & x_2 & \leq & 2 \\ & -x_1 & - & x_2 & \leq & -2 \\ & 4x_1 & + & 6x_2 & \geq & 9 \\ & x_1 & & & \geq & 0. \end{array}$$

Multiply the third constraint by (-1) to reverse the direction of the inequality.

$$\begin{array}{lrcrcl} max & 2x_1 & - & 3x_2 & & \\ s.t. & x_1 & + & x_2 & \leq & 2 \\ & -x_1 & - & x_2 & \leq & -2 \\ & -4x_1 & - & 6x_2 & \leq & -9 \\ & x_1 & & & \geq & 0. \end{array}$$

Finally, since x_2 is a free variable, we replace all its occurrences by $x_2 = u_2 - v_2$ requiring that $u_2 \geq 0$ and $v_2 \geq 0$:

$$\begin{array}{lrcrcrcl} max & 2x_1 & - & 3u_2 & + & 3v_2 & & \\ s.t. & x_1 & + & u_2 & - & v_2 & \leq & 2 \\ & -x_1 & - & u_2 & + & v_2 & \leq & -2 \\ & 4x_1 & + & 6u_2 & - & v_2 & \leq & -9 \\ & x_1 & , & u_2 & , & v_2 & \geq & 0. \end{array}$$

The last problem is equivalent to the original one in the sense that it is infeasible or unbounded if and only if the original one is such and it has an optimal solution if and only if the original one has. Moreover, if (x_1^*, u_2^*, v_2^*) is an optimal solution of the new problem then $(x_1^*, u_2^* - v_2^*)$ is an optimal solution of the original one.

Second approach. *This particular LP problem can be simplified from the very beginning. Since x_2 is a free variable and it appears in the equality constraint $x_1 + x_2 = 2$ we can eliminate it from the problem all together: $x_2 = 2 - x_1$. Substituting it out we obtain*

$$\begin{array}{lrcl} min & -5x_1 & + & 13 \\ s.t. & -2x_1 & \geq & -3 \\ & x_1 & \geq & 0. \end{array}$$

Equivalently

$$\begin{array}{llcl} min & -5x_1 & + & 13 \\ s.t. & x_1 & \leq & 3/2 \\ & x_1 & \geq & 0, \end{array}$$

immediately showing that the optimal solution is $x_1^* = 3/2$ *with optimal value* $-15/2 + 13 = 11/2$. *The optimal value for the variable* x_2 *is of course* $x_2^* = 2 - x_1^* = -7/2$.

The second approach in the example above shows that we need to apply the transformations wisely. Now that we know how to convert any LP problem into standard inequality form, in order to show how to convert any LP problem into standard equality form, all we need to practice is the conversion from SIF into SEF.

Example 2. *Convert into an equivalent LP problem in SEF*

$$\begin{array}{lrcrcr} max & 2x_1 & - & 3x_2 & & \\ s.t. & x_1 & + & x_2 & \leq & 2 \\ & 4x_1 & + & 6x_2 & \leq & -9 \\ & x_1 & , & x_2 & \geq & 0. \end{array}$$

We introduce two new slack variables: one for each constraint. Let $s_1 := 2 - x_1 - x_2$ *and* $s_2 := -9 - 4x_1 - 6x_2$. *The equivalent SEF is*

$$\begin{array}{lrcrcrcrcr} max & 2x_1 & - & 3x_2 & & & & & & \\ s.t. & x_1 & + & x_2 & + & s_1 & & & = & 2 \\ & 4x_1 & + & 6x_2 & & & + & s_2 & = & -9 \\ & x_1 & , & x_2 & , & s_1 & , & s_2 & \geq & 0. \end{array}$$

4 Preprocessing

The standard equality form

$$\left\{ \begin{array}{lrcl} \text{max} & z = cx & & \\ \text{s.t.} & Ax & = & b \\ & x & \geq & 0. \end{array} \right. \tag{3}$$

is used as the starting point of the simplex algorithm that we are going to develop. The functional constraints $Ax = b$ is a system of m linear equations with n unknowns. Since A is an $m \times n$ matrix $\operatorname{rank} A \leq m$. Let us see that without loss of generality we may assume that $\operatorname{rank} A = m$. Indeed, suppose that $\operatorname{rank} A < m$. This means that there is a non-zero linear combination between the rows of A, that is one of the rows, say the first one, is a linear combination of the rest. If the first coordinate of vector b is not equal to the same linear combination of the rest of its coordinates then the system of equations $Ax = b$ is inconsistent, it has no solution. The LP problem is infeasible and there is nothing to solve. On the other hand, if the first coordinate of vector b is equal to the same linear combination of the rest of its coordinates then the first equation in $Ax = b$ is redundant and we may remove it. (The first equation is a linear combination of the rest. Make the distinction between the i-th row of A and the i-th equation in $Ax = b$!) Thus we are left with a system $A'x = b'$ of $m-1$ equations in n unknowns. Notice that rank A'=rank A. We can repeat the procedure until rank A' is equal to the number of equations or find out that the problem is infeasible. Thus, we make the following standing assumption.

Assumption 3. Without loss of generality, we assume that rank $A = m$ in (3).

5 Bases and basic solutions

In order to convert the strawberry model

$$\text{(4)} \qquad \left\{ \begin{array}{lrcrcl} \max & 2x_1 & + & 3x_2 & & \\ \text{s.t.} & x_1 & + & x_2 & \leq & 6 \\ & -x_1 & + & x_2 & \leq & 4 \\ & 2x_1 & + & x_2 & \leq & 10 \\ & x_1 & , & x_2 & \geq & 0 \end{array} \right.$$

into an equivalent one in standard equality form we need to introduce three new slack variables – one for each functional constraint:

$$\text{(5)} \qquad \left\{ \begin{array}{lrcrcrcrcrcl} \max & 2x_1 & + & 3x_2 & & & & & & & & \\ \text{s.t.} & x_1 & + & x_2 & + & x_3 & & & & & = & 6 \\ & -x_1 & + & x_2 & & & + & x_4 & & & = & 4 \\ & 2x_1 & + & x_2 & & & & & + & x_5 & = & 10 \\ & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq & 0. \end{array} \right.$$

The data for this problem is

$$c := (2,3,0,0,0), \; A := \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 1 \end{pmatrix}, \; b := \begin{pmatrix} 10 \\ 6 \\ 4 \end{pmatrix}.$$

By A_j we denote the j-th column in the matrix A, that is, $A = [A_1, ..., A_n]$.

Definition 4. A *basis of the system of equations* $Ax = b$ is an m-element subset B of $\{1, ..., n\}$ such that the set of column vectors $\{A_j \mid j \in B\}$ is linearly independent.

For a given basis B, denote by A_B the $m \times m$ submatrix of A formed by the columns with indexes in B:

$$A_B := [A_j, j \in B].$$

The condition in the definition of a basis means that A_B is non-singular, that is, $\det A_B \neq 0$, that is the rows and the columns of A_B form a basis (in the sense of linear algebra) of $\mathbb{R}^m$. It is very important that one makes a clear distinction between the notions a basis of a vector space (from linear algebra) and a basis of a system of linear equations just introduced.

Example 5. The sets $\{3,4,5\}$ and $\{1,3,4\}$ are bases for the system of functional constraints in (4). Indeed, if $B=\{3,4,5\}$ then $A_B=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$ is clearly non-singular (a permutation of the rows gives the identity matrix). If $B=\{1,3,4\}$ then $A_B=\begin{pmatrix}1&1&0\\-1&0&1\\2&0&0\end{pmatrix}$ is non-singular because $\det A_B=-1\neq 0$.

The variables x_j for $j\in B$ are called *basic variables* while x_j for $j\notin B$ are called *non-basic variables*. Let

$$x_B := (x_j, j\in B)$$

be the subvector of x formed by the basic variables.

Since A_B is non-singular, the system of equations $A_Bx_B=b$ has a unique solution $x_B=A_B^{-1}b$. Notice that $A_Bx_B=b$ is obtained from $Ax=b$ by setting $x_j=0$ for all $j\notin B$. Thus the system

$$\left[\begin{array}{ll} Ax=b & \\ x_j=0 & \text{for all } j\notin B\end{array}\right. \tag{6}$$

has a unique solution, called the *basic solution determined by the basis* B.

Definition 6. A *basic solution of a system of linear equations* $Ax=b$ is the unique solution of (6) for some basis B.

In other words a basic solution of $Ax=b$ is a vector x^* that is a solution to the system of equations and $x_j^*=0$ for all $j\notin B$ for some basis B.

Theorem 7. *Let A be an $m\times n$ matrix of rank m. Let x^* be a solution of $Ax=b$. Then x^* is a basic solution of $Ax=b$ if and only if $\{A_j \mid x_j^*\neq 0\}$ is a linearly independent set.*

Proof. Suppose that x^* is a basic solution of $Ax=b$ determined by the basis B. Then $\{j \mid x_j^*\neq 0\}\subseteq B$. Thus the columns $\{A_j \mid x_j^*\neq 0\}$ are among the columns of A_B which are linearly independent by definition of B.

Suppose now that $\{A_j \mid x_j^*\neq 0\}$ is a linearly independent set of columns. Since the rank $A=m$ we can complete the set $\{A_j \mid x_j^*\neq 0\}$ to a set of m linearly independent columns of A. (Why?) Denote their indexes by B. Check that x^* is the basic solution determined by B. □

According to this theorem, a solution x^* of $Ax=b$ is **not** basic if and only if, $\{A_j \mid x_j^*\neq 0\}$ is a linearly dependent set.

Example 8. For the system of equations:

$$\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & - & x_3 & + & 2x_4 & = & 3,\\ x_1 & + & 2x_2 & + & 2x_3 & - & x_4 & = & 3\end{array}$$

which of the following is a basic solution?

(i) $(2,0,1,1)^T$,

(ii) $(0,1,1,0)^T$

(iii) $(0,0,3,3)^T$,

(iv) $(1,1,0,0)^T$,

(v) $(3,0,0,0)^T$,

In this case $m=2$ and $n=4$, $A = \begin{pmatrix} 1 & 2 & -1 & 2 \\ 1 & 2 & 2 & -1 \end{pmatrix}$, and $b=(3,3)^T$.

(i) Since every basic solution must have at least $n-m$ zeros, vector $(2,0,1,1)^T$ is not one.

(ii) It is not basic solution, because $(0,1,1,0)^T$ is not even a solution to the system of equations $Ax=b$.

(iii) Basic because the third and the fourth column of A are linearly independent, by the theorem since A has rank 2. The basis is $B=\{3,4\}$. (Check!)

(iv) Since the first and the second column from A are not linearly independent, x^* is not a basic solution, by the theorem since A has rank 2.

(v) Since A has rank 2, by the theorem, we have to check if the first column is linearly independent. It is non-zero, so yes. Thus, x^* is a basic solution. In fact there are two bases that have x^* as their corresponding basic solution $B=\{1,3\}$ and $B=\{1,4\}$. (Check.)

One of the observations from the last example is that two different bases may have the same basic solution.

Definition 9. A basic solution x of $Ax=b$ is called *feasible* if $x \geq 0$. If the basic solution determined by a basis B is feasible then B is called a *feasible basis.* A basic solution x of $Ax=b$ is called *degenerate* if x has more than $n-m$ zero entries. If the basic solution determined by a basis B is degenerate then B is called a *degenerate basis.*

5.1 Basic feasible solutions for a system of inequalities

Let

$$Ax \leq b \tag{7}$$

be a system of inequalities. Consider the related system of equations (in $n+m$ variables)

$$(A, I) \begin{pmatrix} x \\ s \end{pmatrix} = b, \tag{8}$$

where I is the $m \times m$ identity matrix. (The rank of the augmented matrix (A, I) is always m, no matter what A is.)

Definition 10. Vector x is called *basic solution* (resp. basic feasible solution) of (7) if $(x, s) := (x, b - Ax)$ is a basic solution (resp. basic feasible solution) of (8).

Notice that a basic solution x^* of $Ax \leq b$ need not be a solution to the system of inequalities (e.g. the points A, D, H, I, J in the example in the next subsection). But the augmented vector $(x, s) := (x, b - Ax)$ is always a solution to (8). By definition, x^* is a basic feasible solution of (7) if and only if $(x, s) := (x^*, b - Ax^*)$ is a basic feasible solution of (8), that is, (x, s) is basic for (8) and $(x, s) \geq 0$ or equivalently $Ax^* \leq b$ and $x^* \geq 0$.

5.2 Geometric representation of the basic (feasible) solutions

In this subsection we are going to identify the basic and the basic feasible solutions of the strawberry problem

$$\left\{\begin{array}{lrcrcl} \max & 2x_1 & + & 3x_2 & & \\ \text{s.t.} & 2x_1 & + & x_2 & \leq & 10 \\ & x_1 & + & x_2 & \leq & 6 \\ & -x_1 & + & x_2 & \leq & 4 \\ & x_1 & , & x_2 & \geq & 0 \end{array}\right. \tag{9}$$

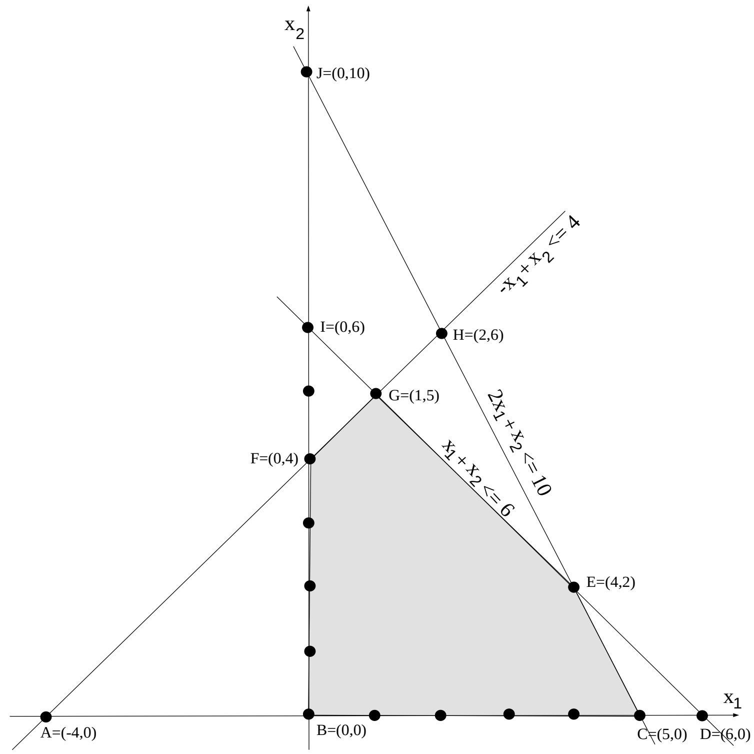
with equivalent standard equality form

$$\left\{\begin{array}{lrcrcccccccl} \max & 2x_1 & + & 3x_2 & & & & & & & & \\ \text{s.t.} & 2x_1 & + & x_2 & + & x_3 & & & & & = & 10 \\ & x_1 & + & x_2 & & & + & x_4 & & & = & 6 \\ & -x_1 & + & x_2 & & & & & + & x_5 & = & 4 \\ & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq & 0. \end{array}\right. \tag{10}$$

The graphical representation of (9) is given below. We saw before that $B = \{3, 4, 5\}$ is a basis of (10) with corresponding basic solution $x^* = (0, 0, 10, 6, 4)^T$. The values of the original variables (x_1, x_2) are $(0, 0)$. This is point B on the figure, the place where the equations $x_1 = 0$ and $x_2 = 0$ intersect. Moreover, notice that x^* is a basic feasible solution and that $(0, 0)$ is a corner point of the feasible region.

Again from before we know that $B = \{1, 3, 4\}$ is a basis of (10) with corresponding basic solution $x^* = (-4, 0, 18, 10, 0)^T$. The values of the original variables (x_1, x_2) are $(-4, 0)$. This is point A on the figure, the place where the equations $-x_1 + x_2 = 4$ and $x_2 = 0$ intersect. Moreover, notice that x^* is not a basic feasible solution and that $(-4, 0)$ is not a point in the feasible region.

One easily sees the pattern: every one of the points A,B,...,J corresponds to one basis of (10). The coordinates of A,B,...,J are the first two coordinates of the basic solution corresponding to that basis. If the basis is feasible then the corresponding point among A,B,...,J is a corner point of the feasible region. If the basis is not feasible the corresponding point among A,B,...,J is not in the feasible region.



For example, let us find the basis corresponding to the point H= $(2,6)$ on the figure. Since $x_1 = 2$ and $x_2 = 6$ we easily find the values of x_3, x_4, and x_5 using using the fact that $(x_1, x_2, x_3, x_4, x_5)$ must satisfy the functional constraints in (10):

$$\begin{aligned} x_3 &= 10 - 2x_1 - x_2 = 0, \\ x_4 &= 6 - x_1 - x_2 = -2, \\ x_5 &= 4 + x_1 - x_2 = 0. \end{aligned}$$

Thus $x^* = (2, 6, 0, -2, 0)^T$ is a solution to $Ax = b$. To see that x^* is a basic solution we need to find a basis B for which x^* is the solution of (6). Since x^* has three non-zero coordinates and the corresponding columns from A are linearly independent (check), by Theorem 7, x^* is a basic solution corresponding to the basis $B = \{1, 2, 4\}$. Notice that x^* is not a basic feasible solution, something that we expected since H in not in the feasible region.

Consider now a slightly modified version of the previous example: include the new constraint

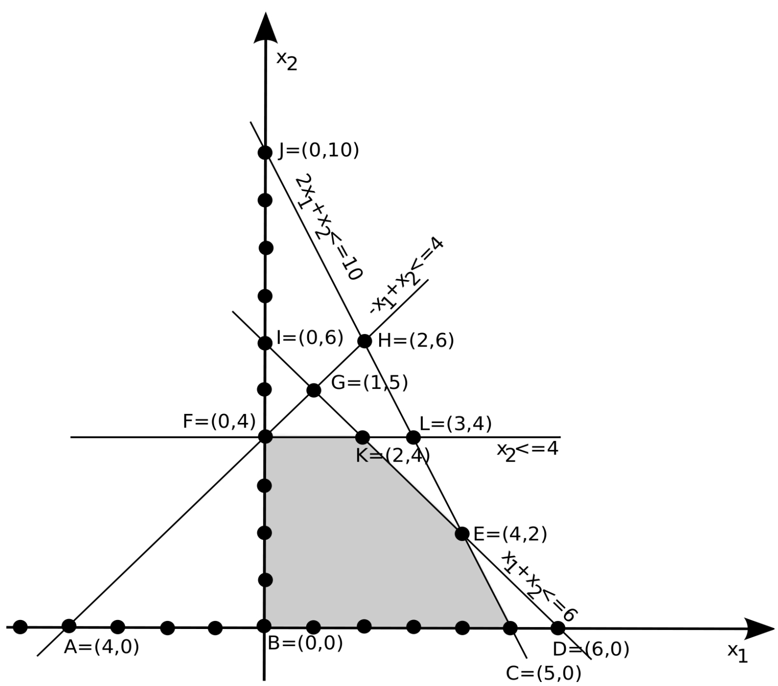
$x_4 \leq 4$:

$$
(11) \qquad \left\{ \begin{array}{lrcrcl}
\max & 2x_1 & + & 3x_2 & & \\
\text{s.t.} & 2x_1 & + & x_2 & \leq & 10 \\
 & x_1 & + & x_2 & \leq & 6 \\
 & -x_1 & + & x_2 & \leq & 4 \\
 & & & x_2 & \leq & 4 \\
 & x_1 & , & x_2 & \geq & 0
\end{array} \right.
$$

with equivalent standard equality form

$$
(12) \qquad \left\{ \begin{array}{lrcrcrcrcrcrcl}
\max & 2x_1 & + & 3x_2 & & & & & & & & & & \\
\text{s.t.} & 2x_1 & + & x_2 & + & x_3 & & & & & & & = & 10 \\
 & x_1 & + & x_2 & & & + & x_4 & & & & & = & 6 \\
 & -x_1 & + & x_2 & & & & & + & x_5 & & & = & 4 \\
 & & & x_2 & & & & & & & + & x_6 & = & 4 \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & , & x_6 & \geq & 0.
\end{array} \right.
$$

Note that now we have four equality constraints with six variables, that is $m = 4$ and $n = 6$. The graphical representation of (11) is given below. Note the now we have three more basic



solutions corresponding to the points F, K, and L. Both F and K correspond to a basic feasible solutions, while L corresponds to a basic solution that is not feasible. Now, the basic feasible solution corresponding to F has a new feature, it is degenerate. Indeed, it is $x = (x_1, ..., x_6) = (0, 4, 6, 2, 0, 0)$ and has more than $n - m = 2$ zeros. Informally, starting from a problem in standard inequality form, basic solutions that are at the "intersection" of strictly more than $n-m$ constraints (here the positivity constraints are also counted as constraints) are degenerate. Note that n is the number of variables in the corresponding standard equality form.

In order to formally justify the observations in this subsection, we need to introduce the notion of a convex set.

6 Convex sets

Let $x, y \in \mathbb{R}^n$ by $[x, y]$ we denote the set $\{(1-\lambda)x + \lambda y \,|\, \lambda \in [0,1]\}$, the line segment between x and y.

Definition 11. *A set $C \subset \mathbb{R}^n$ is convex if for any points $x, y \in C$ we have $[x, y] \subset C$.*

Several examples and simple observations follow.

- The empty set $\emptyset$ is convex.
- The whole space $\mathbb{R}^n$ is convex.
- Subspaces of $\mathbb{R}^n$ are convex.
- The positive orthant $R^n_+ := \{x \in R^n \,|\, x \geq 0\}$ is convex.
- If $a \in R^n$ and $\beta \in R$ then the half space $H^+ := \{x \in \mathbb{R}^n \,| a^T x \leq \beta\}$ is convex. Indeed, let x, y be any two points in H^+ and let λ be any number in $[0,1]$. We have to check that $(1-\lambda)x + \lambda y \in H^+$:
$$a^T((1-\lambda)x + \lambda y) = (1-\lambda)a^T x + \lambda a^T y \leq (1-\lambda)\beta + \lambda\beta = \beta.$$
- If $a \in R^n$ and $\beta \in R$ then the *hyperplane* $H := \{x \in \mathbb{R}^n \,| a^T x = \beta\}$ is convex.
- Let $\mathcal{A}$ be any indexing set and suppose that C_α is a convex set for all $\alpha \in \mathcal{A}$. Then the intersection $C := \cap_{\alpha\in\mathcal{A}} C_\alpha$ is convex. Indeed, let x, y be any two points in C and let λ be any number in $[0,1]$. We have to show that $(1-\lambda)x + \lambda y \in C$. Since for every α, $x, y \in C_\alpha$ and C_α is convex then, $(1-\lambda)x + \lambda y \in C_\alpha$. Therefore $(1-\lambda)x + \lambda y \in \cap_{\alpha\in\mathcal{A}} C_\alpha$.

Recall that the feasible region (FR) of an LP problem in standard inequality form is

$$\begin{aligned} FR &= \{x \in R^n \,|\, Ax \leq b, x \geq 0\} \\ &= \{x \in R^n \,|\, Ax \leq b\} \cap \mathbb{R}^n_+ \\ &= \cap_{j=1,\dots,m} \{x \in R^n \,|\, a_{j1}x_1 + \cdots + a_{jn}x_n \leq b_j\} \cap \mathbb{R}^n_+. \end{aligned}$$

All that shows that FR is the intersection of convex sets and is thus convex. Similarly one shows that the feasible region of any LP problem (not just in standard inequality form) is a convex set.

Proposition 12. *The set of optimal solutions of an LP problem is convex.*

Proof. If the LP problems has no optimal solutions then the set of optimal solutions is empty and thus convex. So suppose now that the LP problem has at least one optimal solution, and let z^* be the optimal value. The set of optimal solutions is then

$$S := \{x \in R^n \mid x \text{ is feasible and } cx = z^*\}.$$

This is just the intersection of the feasible region with the hyperplane $\{x \in \mathbb{R}^n \mid cx = z^*\}$, both convex sets. $\square$

The last proposition shows that for an LP problem exactly one of the following three situations holds 1) it has zero optimal solutions (when it is infeasible or unbounded) 2) it has one optimal solution, 3) it has infinitely many optimal solution (if it has at least two different optimal solutions x and y then every point on the segment $[x, y]$ is an optimal solution as well).

We say that $z \in \mathbb{R}^n$ is a *convex combination* of the vectors $y_1, ..., y_N \in \mathbb{R}^n$ if there are numbers λ_1,...,$\lambda_N \in [0, 1]$ with $\sum_{i=1}^N \lambda_i = 1$ such that

$$z = \lambda_1 y_1 + \cdots + \lambda_N y_N.$$

Thus, a convex combination between several vectors is just a linear combination with the additional requirements that the coefficients are in $[0, 1]$ and sum to 1.

Exercise 13. Show that a subset $C \subseteq \mathbb{R}^n$ is convex if and only if any convex combination between any number of vectors in C is also in C.

6.1 Extreme points of convex sets

Let C be a convex set in $\mathbb{R}^n$.

Definition 14. A point $e \in C$ is called an *extreme point* if it cannot be represented as $e = (1 - \lambda)x + \lambda y$ for some distinct $x, y \in C$ and $\lambda \in (0, 1)$.

Notice that in the definition we require $x \neq y$ and $\lambda \notin \{0, 1\}$. Here are some examples.

- If $C = [0, 1] \subset \mathbb{R}$ then the extreme points of C are $\{0, 1\}$.
- The set $C = (1, 0)$ is convex but has no extreme points.
- For any $a \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$ the half-space H^+ and the hyperplane H have no extreme points, even though they are closed sets (in the sense of real analysis).

In order to formulate a good sufficient condition for a convex set to have an extreme point, we need two notions. A subset of $\mathbb{R}^n$ is called *closed* if it contains all its "boundary" points. More precise definition, given in a real analysis course, is beyond our needs. A subset of $\mathbb{R}^n$ is called *bounded* if it can be enclosed in a ball with large enough radius.

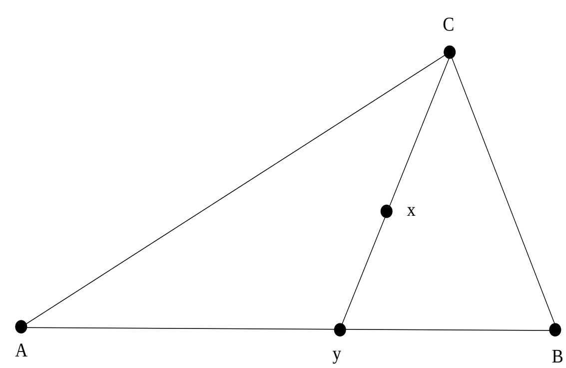
- The set $(0,1) \subset \mathbb{R}$ is not closed but bounded.
- The set $[0,1] \subset \mathbb{R}$ is closed and bounded.
- For any $a \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$ the half-space H^+ and the hyperplane H are closed sets but not bounded.
- For any $a \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$ then set $\{x \in \mathbb{R}^n \mid a^T x < \beta\}$ is not closed and not bounded.

Fact 15. *Intersection of any number of closed sets is closed. Union of finitely many closed sets is closed.*

Just like we showed that the feasible region and the set of optimal solutions of an LP problem are convex sets one can show that they are also closed sets. The following theorem is stated without a proof.

Theorem 16 (Krein-Milman-Caratheodory)**.** *If C is a closed, convex and bounded subset of $\mathbb{R}^n$ then it has an extreme point. Moreover, every point in C can be written as a convex combination of at most $(n+1)$ extreme points.*

Example 17. Consider a point x in a triangle with vertices A, B, C in the plane $\mathbb{R}^2$. Clearly, the vertices are the extreme points of the triangle. How can we express x as a convex combination of the vertices? Let y be the point where the line from C through x intersects the side AB. Since y is



on the segment $[A,B]$ it can be expressed as a convex combination of A and B: $y = (1-\lambda)A + \lambda B$ for some $\lambda \in [0,1]$. Since x is on the segment $[y,C]$ it can be expressed as $x = (1-\mu)y + \mu C$ for some $\mu \in [0,1]$. Substituting the expression for y we get

$$x = (1-\mu)(1-\lambda)A + (1-\mu)\lambda B + \mu C.$$

This expresses x as a convex combination of A, B, C (show that this is indeed a convex combination). Now, even if x is not inside a triangle, but say, inside a convex hexagon, or octagon, by the Krein-Milman-Caratheodory theorem, it can be expressed as a convex combination of not more than three of the vertices (how?).

The relevance of all that to linear programming is given by the following theorem.

Theorem 18. *Let A be an $m \times n$ matrix with rank m, and let $FR := \{x \mid Ax = b,\ \ x \geq 0\}$. Then x^* is a basic feasible solution of $Ax = b$ if and only if x^* is an extreme point of FR.*

Proof. Because the theorem is an "if an only if" statement, we have to prove two things:

(i) If x^* is a basic feasible solution then it is an extreme point of FR, and conversely

(ii) If x^* is an extreme point of FR then it is a basic feasible solution of $Ax = b$, $x \geq 0$.

(i) Suppose that x^* is a basic feasible solution of $Ax = b$. Then $Ax^* = b$ and $x^* \geq 0$ showing that $x^* \in FR$. In addition, there is a basis B such that x^* is the unique solution of

$$\left[\begin{array}{l} Ax = b \\ x_j = 0 \quad \text{for all } j \notin B. \end{array}\right. \tag{13}$$

We will show that it is an extreme point of FR by assuming the contrary and reaching a contradiction. Thus, suppose that x^* is not an extreme point of FR. Then there exist two distinct points x^1 and x^2 in FR and a $\lambda \in (0, 1)$ such that $x^* = \lambda x^1 + (1 - \lambda)x^2$. But x^* is a basic feasible solution corresponding to B, this means that the non-basic entries of x^* are zero:

$$\text{if } j \notin B \text{ then } 0 = x_j^* = \lambda x_j^1 + (1 - \lambda)x_j^2.$$

The right-hand side is the sum of two positive numbers: λx_j^1 and $(1 - \lambda)x_j^2$. For it to be zero we must have $\lambda x_j^1 = (1 - \lambda)x_j^2 = 0$. Since $0 < \lambda < 1$ we must have $x_j^1 = x_j^2 = 0$. This is true for every $j \notin B$. Thus, x^1 and x^2 are both solutions of the system of equations (13) or $x^* = x^1 = x^2$. This is a contradiction with the fact that x^1 and x^2 are distinct. Therefore x^* is an extreme point.

(ii) Suppose now, that x^* is an extreme point of FR (so $x^* \in FR$ and thus it satisfies $Ax^* = b$ and $x^* \geq 0$). We will show that x^* is a basic solution of $Ax = b$. Suppose that this is not the case (we will reach a contradiction). Then

$$\{A_j \mid x_j^* \neq 0\} = \{A_j \mid x_j^* > 0\}$$

is a linearly dependent set. Define the set of indexes $J = \{j \mid x_j^* > 0\}$. There exist y_j, $j \in J$, not all zero such that

$$\sum_{j \in J} A_j y_j = 0.$$

Now define $y_j = 0$ for all $j \in \{1, 2, ..., n\} \backslash J$ and let $y := (y_1, ..., y_n)^T$. So $y \in \mathbb{R}^n$ and $y \neq 0$. Moreover

$$Ay = 0,$$

and $y_j = 0$ whenever $x_j^* = 0$. Consider the points $x^1 = x^* - \epsilon y$, $x^2 = x^* + \epsilon y$, where $\epsilon > 0$. Notice that, for ϵ close enough to zero, $x^1, x^2 \geq 0$. Also,

$$Ax^1 = Ax^* - \epsilon Ay = Ax^* = b,$$

and similarly $Ax^2 = b$. Therefore, $x^1, x^2 \in FR$. The fact that $y \neq 0$ implies that $x^1 \neq x^2$. Finally, notice that

$$x^* = \frac{1}{2}x^1 + \frac{1}{2}x^2.$$

This is a contradiction to the fact that x^* is an extreme point of FR. Therefore x^* must be a basic feasible solution, as required. $\square$

Analogous theorem holds the feasible region of an LP problem in standard inequality form. Notice that we do not need to require that the matrix A have rank m. (Why?)

Theorem 19. *Let A be an $m \times n$ matrix and let $FR := \{x \mid Ax \leq b, \ \ x \geq 0\}$. Then x^* is a basic feasible solution of $Ax \leq b$ if and only if x^* is an extreme point of FR.*

Hence by the Krein-Milman-Caratheodory if the feasible region (resp. the set of optimal solutions) of an LP problem is bounded, then every feasible solution (resp. every optimal solution) is a convex combination of at most $(n + 1)$ basic feasible solutions.

7 Bases and their tableaux

The simplex method works with LP problems in standard equality form:

$$\text{(14)} \qquad \begin{cases} \max & z = cx \\ \text{s.t.} & \quad Ax \ = \ b \\ & \quad\ \ x \ \geq \ 0. \end{cases}$$

Let B be a basis of $Ax = b$ with corresponding basic solution x^*. Define the set $N := \{1, 2, ..., n\} \backslash B$ of all indexes that are not in the basis. Recall that $x = (x_B, x_N)$ is the partitioning of x into basic and non-basic variables. The goal of this section is to describe the *tableau* corresponding to the basis B. First we state the formal procedure and then we illustrate it with an example. It helps to read the following three steps and the example at the same time.

1) Since the matrix A_B is non-singular, we can apply elementary row operations to the equations $Ax = b$ until we obtain an equivalent system of equations $\bar{A}x = \bar{b}$ in which the columns of $\bar{A}$ corresponding to the basic variables form an $m \times m$ identity matrix. In other words, the final result of these elementary row operations is equivalent to multiplying the system $Ax = b$ on the left by A_B^{-1}: $A_B^{-1}Ax = A_B^{-1}b$, that is $\bar{A} := A_B^{-1}A$ and $\bar{b} := A_B^{-1}b$.

2) Solve each equation in $\bar{A}x = \bar{b}$ for its basic variable and substitute it out of the objective function $z = cx$ to obtain the equivalent objective function $z = \bar{c}x + \bar{v}$.

3) Finally, write $z = \bar{c}x + \bar{v}$ as $z - \bar{c}x = \bar{v}$.

4) We keep in mind that all variables have to be non-negative but for simplicity we do not write it.

The resulting representation of the LP problem (14) is called the tableau corresponding to the basis B.

Example 20. Consider the strawberry example in standard equality form.

$$
\left\{
\begin{array}{lrcrcrcrcrcl}
\max & z = 2x_1 & + & 3x_2 & & & & & & & & \\
\text{s.t.} & 2x_1 & + & x_2 & + & x_3 & & & & & = & 10 \\
 & x_1 & + & x_2 & & & + & x_4 & & & = & 6 \\
 & -x_1 & + & x_2 & & & & & + & x_5 & = & 4 \\
 & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & \geq & 0.
\end{array}
\right.
\tag{15}
$$

Let the basis B be $\{3, 4, 5\}$. The equations are already in the form that we want: the columns with indexes 3, 4 and 5 form a 3×3 identity matrix. There is nothing to do in step 2) because there are no basic variables in the objective function. Thus, the tableau corresponding to this basis is

$$
\begin{array}{lrcrcrcrcrcl}
\max & z - 2x_1 & - & 3x_2 & & & & & & & = & 0 \\
\text{s.t.} & 2x_1 & + & x_2 & + & x_3 & & & & & = & 10 \\
 & x_1 & + & x_2 & & & + & x_4 & & & = & 6 \\
 & -x_1 & + & x_2 & & & & & + & x_5 & = & 4
\end{array}
$$

We know that $B = \{3, 4, 1\}$ is also a basis for (15). After elementary row operations on the equality constraints we can write them as

$$
\begin{array}{lrcrcrcrcrcl}
\max & z = 2x_1 & + & 3x_2 & & & & & & & & \\
\text{s.t.} & & + & 3x_2 & + & x_3 & & & + & 2x_5 & = & 18 \\
 & & + & 2x_2 & & & + & x_4 & + & x_5 & = & 10 \\
 & x_1 & - & x_2 & & & & & - & x_5 & = & -4.
\end{array}
$$

Notice that columns with indexes $3, 4$ and 1 form a 3×3 identity matrix. Now we want to eliminate the basic variables from the objective function. In this case, there is only one basic variable in the objective function: x_1. From the third equation we solve for x_1 and substitute it out of the *z-row*:

$$
\begin{array}{lrcrcrcrcrcl}
\max & z = & & 5x_2 & & & & & + & 2x_5 & - & 8 \\
\text{s.t.} & & + & 3x_2 & + & x_3 & & & + & 2x_5 & = & 18 \\
 & & + & 2x_2 & & & + & x_4 & + & x_5 & = & 10 \\
 & x_1 & - & x_2 & & & & & - & x_5 & = & -4.
\end{array}
$$

Finally, the tableau corresponding to the basis $B = \{3, 4, 1\}$ is

$$
\begin{array}{lrcrcrcrcrcl}
\max & z & - & 5x_2 & & & & & - & 2x_5 & = & -8 \\
\text{s.t.} & & + & 3x_2 & + & x_3 & & & + & 2x_5 & = & 18 \\
 & & + & 2x_2 & & & + & x_4 & + & x_5 & = & 10 \\
 & x_1 & - & x_2 & & & & & - & x_5 & = & -4.
\end{array}
\tag{16}
$$

To summarize, the tableau of (14) corresponding to a basis B is obtained from

$$
\left[
\begin{array}{rcl}
z - cx & = & 0 \\
Ax & = & b
\end{array}
\right.
$$

by applying elementary row operations until the following form is reached

$$\left[\begin{array}{rcl} z - \sum_{j\in N} \bar{c}_j x_j & = & \bar{v} \\ x_i + \sum_{j\in N} \bar{a}_{ij} x_j & = & \bar{b}_i, \ \ \forall i \in B. \end{array}\right.$$

For example, in tableau (16) corresponding to the basis $B = \{3,4,1\}$ we have $\bar{c}_2 = +5$, $\bar{c}_5 = +2$, $\bar{v} = -8$, $\bar{a}_{32} = 3$, $\bar{a}_{35} = 2$, $\bar{a}_{31} = 0$, $\bar{b}_3 = 18$. Coefficients like $\bar{b}_2$, $\bar{b}_5$, $\bar{a}_{13}$, $\bar{a}_{31}$,... are not defined for that basis. In other words, in the tableau corresponding to B, we label the rows of the matrix $\bar{A}$ and the entries of the vector $\bar{b}$ by the elements of B. It is a bit strange at first, but one quickly gets used to that. The row $x_i + \sum_{j\in N} \bar{a}_{ij} x_j = \bar{b}_i$ in the tableau is called the x_i-row, or just the i-row, where $i \in B$, and the row $z - \sum_{j\in N} \bar{c}_j x_j = \bar{v}$ is called the z-row. The coefficients $\bar{c}_j$ for $j \in N$ of the z-row are called *reduced costs*, or $\bar{c}_j$ is the *reduced cost of the variable* x_j.

The tableau corresponding to a basis B contains all the information that the original problem has (elementary row operations do not change a system of equations). But the nice feature of a tableau is that it displays clearly information about the basis B:

1) The basic solution corresponding to the basis B is $x_B = \bar{b}$ and $x_N = 0$.

2) The objective value of that basic solution is $\bar{v}$.

Thus, if the right-hand side $\bar{b}$ of the tableau is $\bar{b} \geq 0$ the the basic solution is feasible and the basis B is feasible.

7.1 Matrix form of the tableau

Given an LP problem in standard equality form (14) and a basis B, the goal now is to derive an expression for the tableau corresponding to B in matrix notation. We have to do essentially two things: 1) Solve $Ax = b$ for the basic variables, and 2) eliminate the basic variables from $z = cx$. We formally split all the data into basic and non-basic parts:

$$c = (c_B, c_N), \ \ x = (x_B, x_N)^T, \ \ A = (A_B, A_N).$$

1) Thus the system $Ax = b$ is the same as

$$(A_B, A_N) \begin{pmatrix} x_B \\ x_N \end{pmatrix} = b \tag{17}$$

or after multiplying, to

$$A_B x_B + A_N x_N = b.$$

Since A_B is invertible, we may multiply the last equation by A_B^{-1} on the left to obtain

$$x_B + A_B^{-1} A_N x_N = A_B^{-1} b. \tag{18}$$

With one swoop we solved the system of equations for the basic variables x_B:

$$x_B = A_B^{-1} b - A_B^{-1} A_N x_N.$$

2) Substituting out the basic variables from the objective function we obtain

$$\begin{aligned} z = cx = (c_B, c_N)\begin{pmatrix} x_B \\ x_N \end{pmatrix} &= c_B x_B + c_N x_N \\ &= c_B(A_B^{-1}b - A_B^{-1}A_N x_N) + c_N x_N \\ &= c_B A_B^{-1}b - c_B A_B^{-1}A_N x_N + c_N x_N \\ &= (c_N - c_B A_B^{-1}A_N)x_N + c_B A_B^{-1}b. \end{aligned}$$

Thus the z-row is

$$z - (c_N - c_B A_B^{-1}A_N)x_N = c_B A_B^{-1}b.$$

We are done. In matrix notation, the tableau corresponding to the basis B is

$$\left[\begin{array}{rcl} z - (c_N - c_B A_B^{-1}A_N)x_N & = & c_B A_B^{-1}b \\ x_B + A_B^{-1}A_N x_N & = & A_B^{-1}b. \end{array}\right.$$

As a bonus, we obtained matrix formulae for our previous notation

$$\begin{aligned} \bar{c} &= c_N - c_B A_B^{-1}A_N \\ \bar{v} &= c_B A_B^{-1}b \\ \bar{A} &= (I, A_B^{-1}A_N) \\ \bar{b} &= A_B^{-1}b. \end{aligned}$$

8 Modeling Examples

8.1 Currency Arbitrage Model

Problem. A company has a capital of 5 million dollars with which they want to play on the currency market for US dollars (\$), Euros (€), British Pounds (£), Japanese Yen (¥) and Mexican Pesos (₱). The currency dealers set the following limitations:

a transaction in \$ cannot be larger than 5 million \$
a transaction in € cannot be larger than 3 million €
a transaction in £ cannot be larger than 3.5 million £
a transaction in ¥ cannot be larger than 100 million ¥
a transaction in ₱ cannot be larger than 2.8 million ₱

You may borrow funds from the bank in any currency and no interest will be incurred on them as long as they are returned within an hour. The table below shows the current exchange rates between the currencies. For example, 1 US Dollar is equal to 0.769 Euros.

	\$	€	£	¥	₱
\$	1	0.769	0.625	105	0.342
€	$\frac{1}{0.769}$	1	0.813	137	0.445
£	$\frac{1}{0.625}$	1/0.813	1	169	0.543
¥	$\frac{1}{105}$	$\frac{1}{137}$	$\frac{1}{169}$	1	0.0032
₱	$\frac{1}{0.342}$	$\frac{1}{0.445}$	$\frac{1}{0.543}$	$\frac{1}{0.0032}$	1

Is it possible to increase the US dollar capital of the company by placing several instantaneous trades on these currencies?

Model. For convenience enumerate the currencies as (\$,€,£,¥, ₱) $= (1, 2, 3, 4, 5)$. Let x_{ij} be the amount in currency i used to buy currency j. Since we do not have any other currency than US dollars, if we want to convert x British pounds into Japanese yen, we will have to borrow the pounds from the bank.

The limitations posed by the currency dealers imply the constraints

$$
\begin{aligned}
x_{1j} &\leq 5, \quad j = 2, 3, 4, 5 \\
x_{2j} &\leq 3, \quad j = 1, 3, 4, 5 \\
x_{3j} &\leq 3.5, \quad j = 1, 2, 4, 5 \\
x_{4j} &\leq 100, \quad j = 1, 2, 3, 5 \\
x_{5j} &\leq 2.8, \quad j = 1, 2, 3, 4.
\end{aligned}
$$

We want to maximize the US dollar capital of the company. That is

$$\max 5 - x_{12} - x_{13} - x_{14} - x_{15} + \frac{1}{0.769}x_{21} + \frac{1}{0.625}x_{31} + \frac{1}{105}x_{41} + \frac{1}{0.324}x_{51}.$$

Since we have to return all the money that we borrowed from the bank in the currency that we borrowed it, we need to balance the transactions for every currency. This gives five additional constraints

dollars: $x_{12} + x_{13} + x_{14} + x_{15} - \frac{1}{0.769}x_{21} - \frac{1}{0.625}x_{31} - \frac{1}{105}x_{41} - \frac{1}{0.324}x_{51} \leq 5$

euros: $x_{21} + x_{23} + x_{24} + x_{25} - 0.769x_{12} - \frac{1}{0.813}x_{32} - \frac{1}{137}x_{42} - \frac{1}{0.445}x_{52} = 0$

pounds: $x_{31} + x_{32} + x_{34} + x_{35} - 0.625x_{13} - 0.813x_{23} - \frac{1}{169}x_{43} - \frac{1}{0.543}x_{53} = 0$

yen: $x_{41} + x_{42} + x_{43} + x_{45} - 105x_{14} - 137x_{24} - 169x_{34} - \frac{1}{0.0032}x_{54} = 0$

pesos: $x_{51} + x_{52} + x_{53} + x_{54} - 0.342x_{15} - 0.445x_{25} - 0.543x_{35} - 0.0032x_{45} = 0.$

Finally, we do not want to buy any currency with negative amount of money, that is $x_{ij} \geq 0$ $i, j = 1, ..., 5$. Notice that the zero vector is always a feasible solution to this problem with objective value 5. (This solution corresponds to the situation when we do not take any action.) Notice also that maximizing the objective function is the same as minimizing

$$-x_{12} - x_{13} - x_{14} - x_{15} + \frac{1}{0.769}x_{21} + \frac{1}{0.625}x_{31} + \frac{1}{105}x_{41} + \frac{1}{0.324}x_{51},$$

which is the left-hand side of the dollar balancing constraint. Thus, the optimal value will always be ≥ 5 and the dollar balancing constraint will be satisfied by the optimal solution of the modified LP problem that do not take the dollar balancing constraint into consideration.

If we use a computer package to solve the problem we get that the optimal value is 5.086976255 million dollars. The optimal solution is (all amounts are in millions)

```
x[i,j] =
       1       2            3      4         5
1      0       1.46302      0      0         5
2      0       0            0      0         3
3      3.5     0.930841     0      0         1.31676
4      100     100          100    0         0
5      0       0            2.8    0.96      0
```

Remember that all orders for the transactions have to be submitted to the trader at once and then you should return the loans to the bank as soon as possible to avoid interest. We see that if we follow the investment strategy we will increase the dollar holdings of the company by $\$86,976.255$ dollars. Not bad for a few seconds work.

8.2 Sharing a snow shovel

Three neighbours, Al, Bal, and Cal, want to clear yesterday's snow from around their houses as quickly as possible, with the aid of their jointly owned snowblower. For a snowfall of this size, they could shovel their own places in 3, 4, and 5 hours respectively. Using the snowblower they would take 1,3, and 2 hours, respectively. Formulate, but do not solve, a linear program (and then put it in standard inequality form) whose solution describes how long each should use the snowblower and how long each should shovel so that the last person to finish cleaning his snow finishes as soon as possible. Each person clears only their own place. (Your program should not take any advantage of the particular rates listed. Define the variables you use explicitly.)

Solution: Let x_a, x_b, and x_c denote the amount of time Al, Bal, and Cal shovel by hand respectively.

Let y_a, y_b, and y_c denote the amount of time Al, Bal, and Cal use the snowblower respectively.

Let t be the time when the last one finishes their job.

The LP problem is:

$$\begin{array}{llllllllllllllll}
\min & t & & & & & & & & & & & & & & \\
\text{s.t.} & x_a & + & y_a & & & & & & & & & \leq & t & & \\
 & & & & & x_b & + & y_b & & & & & \leq & t & & \\
 & & & & & & & & & x_c & + & y_c & \leq & t & & \\
 & \frac{1}{3}x_a & + & y_a & & & & & & & & & = & 1 & & \\
 & & & & & \frac{1}{4}x_b & + & \frac{1}{3}y_b & & & & & = & 1 & & \\
 & & & & & & & & & \frac{1}{5}x_c & + & \frac{1}{2}y_c & = & 1 & & \\
 & & & y_a & & & + & y_b & & & + & y_c & \leq & t & & \\
 & x_a & , & y_a & , & x_b & , & y_b & , & x_c & , & y_c & , & t & \geq & 0
\end{array}$$

Explanations:

- The objective is to minimize the time when the last one finishes their job. Thus we have to minimize t.
- To total amount of time that Al spends working should be less than or equal to the time when the last one finishes. This applies to Bal and Cal and explains the first three constraints.
- If Al shovel by hand his place for 3 hours his speed will be $\frac{1}{3}$ of the job per hour. Similarly his speed with the snowblower is 1 of the job per hour. But he is shovelling x_a hours by hand and y_a hours with the snowblower and for that time he finishes the whole job, 1. Thus, we need to have $\frac{1}{3}x_a + y_a = 1$. Similarly for Bal and Cal. This accounts for the last three constraints.
- The last constraint forbids the simultaneous use of the snowblower.
- Finally, all the times should be nonnegative.

If you wish, you may put this LP into an equivalent form as follows:

$$\begin{array}{llllllllllllllll}
\max & & & & & & & & & & & & - & t & & \\
\text{s.t.} & x_a & + & y_a & & & & & & & & & - & t & \leq & 0 \\
 & & & & & x_b & + & y_b & & & & & - & t & \leq & 0 \\
 & & & & & & & & & x_c & + & y_c & - & t & \leq & 0 \\
 & \frac{1}{3}x_a & + & y_a & & & & & & & & & & & = & 1 \\
 & & & & & \frac{1}{4}x_b & + & \frac{1}{3}y_b & & & & & & & = & 1 \\
 & & & & & & & & & \frac{1}{5}x_c & + & \frac{1}{2}y_c & & & = & 1 \\
 & & & y_a & & & + & y_b & & & + & y_c & - & t & \leq & 0 \\
 & x_a & , & y_a & , & x_b & , & y_b & , & x_c & , & y_c & , & t & \geq & 0
\end{array}$$

8.3 A Sample Network Optimization Problem

Example. The Kitty Railroad is in the process of planning relocations of freight cars among the 5 regions of the country to get ready for the fall harvest. The following table shows the cost of moving a car between each pair of regions, along with the number of cars in each at present and the number needed for harvest shipping.

	Region				
From	1	2	3	4	5
1	-	13	9	15	37
2	13	-	14	7	51
3	9	14	-	10	26
4	15	7	10	-	20
5	37	51	26	20	-
Present	113	382	415	480	610
Need	180	505	810	190	310

We want to choose a reallocation plan to get the required number of cars in each region at minimum total moving cost. Write a linear model for this allocation problem.

Let x_{ij} denote the number of cars moved from region i to region j.

Let c_{ij} denote the cost of moving a car from i to j.

Let p_j denote the number of cars presently at j.

Let n_j denote the number of cars needed at j.

$$\text{minimize} \quad \sum_{i=1}^{5} \sum_{\substack{j=1 \\ j\neq i}}^{5} c_{ij} x_{ij}$$

subject to

$$\sum_{\substack{i=1 \\ i\neq k}} x_{i,k} - \sum_{\substack{j=1 \\ j\neq k}} x_{k,j} = n_k - p_k, \quad k = 1, ..., 5$$

$$x_{ij} \geq 0, \text{ for all } i, j = 1, ..., 5,\ i \neq j.$$

There are five constraints, one for each $k = 1, ...5$. Each constraint makes sure that the cars moved into region k minus the cars moved out of the region k must be equal to the *net* need of that region.

8.4 Example: scheduling production and inventory

Consider a factory producing a commodity. The demand for the commodity fluctuates from month to month in a predictable manner over a period of n months. The demand for the commodity in month j will be d_j units, $j = 1, ..., n$. The factory has two shifts: regular and overtime. In order to meet the demand the factory can do three things:

(i) Increase regular production but only up to r units a month.

(ii) Increase overtime production but only up to s units a month.

(iii) Store present excess to cover future shortages.

The cost of

- regular production is a dollars per unit;
- overtime production is b dollars per unit;
- storage is c dollars per unit per month.

Meet the fluctuating demand at the least possible cost.

Solution: For simplicity we will assume that there are only 4 months to plan ahead. The general situation is completely analogous.

In every month j, there are two different actions that can be taken: increase regular or overtime production. Also in every month, except the last one, we may decide to store some excess for later. Thus, we define three variables:

- $x_{j,a}$ - the amount of the commodity produced during regular production cycle, $j = 1, 2, 3, 4$.
- $x_{j,b}$ - the amount of the commodity produced overtime, $j = 1, 2, 3, 4$.
- $x_{j,c}$ - the amount of the commodity stored in month j for month $j + 1$, $j = 1, 2, 3$.

Thus, there are 11 variables. The linear model is

$$\begin{array}{ll}
\text{minimize} & a(x_{1,a} + x_{2,a} + x_{3,a} + x_{4,a}) + b(x_{1,b} + x_{2,b} + x_{3,b} + x_{4,b}) + c(x_{1,c} + x_{2,c} + x_{3,c}) \\
\text{subject to} & 0 \leq x_{j,a} \leq r,\ j = 1, 2, 3, 4, \\
& 0 \leq x_{j,b} \leq s,\ j = 1, 2, 3, 4, \\
& 0 \leq x_{j,c},\ j = 1, 2, 3, \\
& x_{1,a} + x_{1,b} - x_{1,c} = d_1, \\
& x_{2,a} + x_{2,b} + x_{1,c} - x_{2,c} = d_2, \\
& x_{3,a} + x_{3,b} + x_{2,c} - x_{3,c} = d_3, \\
& x_{4,a} + x_{4,b} + x_{3,c} = d_4.
\end{array}$$

8.5 The SUDOKU puzzle

The game Sudoku is a logic-based placement puzzle. One has to enter a digit from 1 to 9 in each cell of a 9×9 grid made up of 9 3×3 subgrids. Initially, some cells have already been assigned a digit. The goal is to fill the remaining cells so that every row, every column and every 3×3 subgrid contains every digit exactly once. A *proper puzzle* is one that has a unique solution. Only proper puzzles are considered in newspapers and magazines. The Sudoku puzzle was invented in Indianapolis in 1979, you guessed by a mathematician, but reached widespread international popularity only in 2005 after being launched by the British newspapers "The Times" at the end of 2004. Here is one difficult example:

	5							1
			5			8	2	
		4		8				9
7			4		3			
		6				3		
			1		2			7
9				5		6		
	8	3			9			
2							1	

In the above puzzle there are 24 filled (given) cells. It is clear that the more the given numbers the easier the puzzle and vice versa. A very good question is: What is the smallest number so that a puzzle with that many given cells is proper, that is, has a unique solution. No body knows the answer to this problem! There are known proper Sudoku puzzles with only 17 filled cells. Needless to say they are extremely difficult to solve by hand. Thus, 17 is an upper bound for the answer in the conjecture. It is proposed (but not proved) that 17 is the answer to the above open problem, that is, every Sudoku puzzle with 16 filled cells must have more than one solution. It is not too difficult to see that if a Sudoku puzzle has 8 filled cells then it has more than one solution, thus 8 is a lower bound for the conjecture. Can you see that?

We can create an LP problem that solves a given Sudoku puzzle. The variables will be $x_{i,j,k}$ where the indexes $i, j, k \in \{1, 2, ..., 9\}$. We define them to be

$$x_{i,j,k} = \begin{cases} 1 & \text{if cell } (i,j) \text{ has value } k \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

The constraints arise from the fact that every row, column and subgrid must contain every number $\{1, 2, ..., 9\}$ exactly once. Do not forget that also every cell of the grid can hold only one number. Let the cells in the i-th subgrid $i = 1, 2, ..., 9$ be denoted by SG_i. For example:

$$SG_1 = \{(1,1), (1,2), (1,3), (2,1)..., (3,3)\}.$$

Now, we can formulate the constraints.

cell: In every cell (i,j) we have to have one number: $\sum_{k=1}^{9} x_{i,j,k} = 1$.

row: In every row i the number k has to appear only once: $\sum_{j=1}^{9} x_{i,j,k} = 1$.

column: In every column j the number k has to appear only once: $\sum_{i=1}^{9} x_{i,j,k} = 1$.

subgrid: In subgrid l the number k has to appear only once: $\sum_{(i,j)\in SG_l} x_{i,j,k} = 1$.

What is the objective function? Well, there is none. All we want is a feasible solution. Thus, any objective function will do! As we will see later in the course, the algorithms for linear programming try to find a feasible solution first and only then they try to find a feasible solution that is optimal. Thus, we can simply choose to maximize $x_{1,1,1}$. It doesn't matter what objective function you choose.