STAT 9657 — Problem Set 1

You may use any results from the course notes when solving the following problems.

(i) Consider the sequence $\{a_n\}$ on $[0,1]$ defined as follows: $a_1 = 0, a_2 = 1, a_2 = 1/2, a_4 = 1/4, a_5 = 3/4, a_6 = 1/8, a_7 = 3/8, a_8 = 5/8, a_9 = 7/8, ...$ What are the limit points of $\{a_n\}$, limsup and liminf?

(ii) Consider the sequence $\{a_n\}$ on $[0,1]$ defined as follows: $a_1 = 0, a_2 = 1, a_2 = 0, a_4 = 1/2, a_5 = 1, a_6 = 0, a_7 = 1/4, a_8 = 2/4, a_9 = 3/4, a_{10} = 1, a_{11} = 0, a_{12} = 1/8, a_{13} = 2/8, a_{14} = 3/8, a_{15} = 4/8, a_{16} = 5/8, a_{17} = 6/8, a_{18} = 7/8, a_{19} = 1, ...$ What are the limit points of $\{a_n\}$, limsup and liminf?

(iii) For any functions $f(x)$ and $g(x)$ defined on (a, ∞) we have

$$\text{limsup}_{x\to\infty}(f(x) + g(x)) \geq \text{limsup}_{x\to\infty} f(x) + \text{liminf}_{x\to\infty} g(x) \geq \text{liminf}_{x\to\infty}(f(x) + g(x)).$$

(iv) Let $f(x)$ and $g(x)$ be any functions defined on (a, ∞) and suppose that

$$\text{liminf}_{x\to\infty}(g(x) - f(x)) \geq 0.$$

Show that

$$\begin{aligned} \text{limsup}_{x\to\infty} f(x) &\leq \text{limsup}_{x\to\infty} g(x), \quad \text{and} \\ \text{liminf}_{x\to\infty} f(x) &\leq \text{liminf}_{x\to\infty} g(x). \end{aligned}$$

(v) Give an example of two sequences $\{a_n\}$ and $\{b_n\}$ such that

$$\begin{aligned} -\infty < \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n < \liminf_{n\to\infty}(a_n + b_n) < \liminf_{n\to\infty} a_n + \limsup_{n\to\infty} b_n \\ < \limsup_{n\to\infty}(a_n + b_n) < \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n < \infty. \end{aligned}$$

(vi) Let X and Y be sets, and let $f : X \to Y$ be a function. Let A_i, $i \in I$ be a family of subsets of X where I is an index set. Let B_j, $j \in J$ be a family of subsets of Y where J is an index set. Carefully show that

(a) $f\Big(\bigcup_{i\in I} A_i\Big) = \bigcup_{i\in I} f(A_i)$;

(b) $f\Big(\bigcap_{i\in I} A_i\Big) \subseteq \bigcap_{i\in I} f(A_i)$;

(c) $f^{-1}\Big(\bigcup_{j\in J} B_j\Big) = \bigcup_{j\in J} f^{-1}(B_j)$;

(d) $f^{-1}\Big(\bigcap_{j\in J} B_j\Big) = \bigcap_{j\in J} f^{-1}(B_j)$;

(vii) Let X, Y, and Z be sets and let $f : X \to Y$, $g : Y \to Z$ be two functions. Carefully show that

(a) $(f^{-1} \circ f)(A) \supseteq A$;

(b) $(f \circ f^{-1})(B) \subseteq B$;

(c) $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$.

(viii) Let $\{a_n\}$ be a convergent sequences of real numbers with $\lim_{n\to\infty} a_n = a$. Define $b_n := \frac{1}{n}\sum_{i=1}^{n} a_i$ for $n = 1, 2, 3, ...$ Show that $\lim_{n\to\infty} b_n = a$.

(ix) For any sets $A_1, A_2, ... \subseteq \Omega$ show that

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c \quad \text{and} \quad \left(\bigcap_{i=1}^{\infty} A_i\right)^c = \bigcup_{i=1}^{\infty} A_i^c.$$

(x) Let $k \in \mathbb{N}$ be a fixed natural number, find the limit of the sequence $\{b_n\}$, where

$$b_n := \frac{1}{n^{k+1}} \sum_{m=1}^{n} m^k.$$

(xi) Let Ω_1 and Ω_2 be two sets and let $f : \Omega_1 \to \Omega_2$ be a function between them. Show that for any σ-algebra $\mathcal{F}$ on Ω_2, we have that $f^{-1}(\mathcal{F})$ is a σ-algebra on Ω_1.

Here, by $f^{-1}(\mathcal{F})$ we understand the collection of all subsets of Ω_1 that are the pre-image of a set in $\mathcal{F}$. In other words:

$$f^{-1}(\mathcal{F}) := \{f^{-1}(A) : A \in \mathcal{F}\}.$$

(xii) Let Ω_1 and Ω_2 be two sets and let $f : \Omega_1 \to \Omega_2$ be a function between them. Suppose that $\mathcal{M}$ is a collection of subsets of Ω_2. Show that $f^{-1}(\sigma(\mathcal{M})) = \sigma(f^{-1}(\mathcal{M}))$.

The following problem is for the diehards and it is not required for the exams.

(xii) What are the limit points of the sequence $\{\sin(n)\}_{n=1}^{\infty}$.