STAT 9657 — Problem Set 10

You may use any results from the course notes when solving the following problems.

For all of the following problems, assume that $X, Y : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ are random variables that are either both non-negative or both integrable. Let $\mathcal{G} \subseteq \mathcal{F}$.

(i) $\mathbb{E}(\mathbb{E}(X|\mathcal{G})) = \mathbb{E}(X)$

(ii) If X is $\mathcal{G}$-measurable, then $\mathbb{E}(X|\mathcal{G}) = X$ a.s.

(iii) If $X = Y$ a.s. then $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(Y|\mathcal{G})$ a.s.

(iv) If $X = c$ a.s. then $\mathbb{E}(X|\mathcal{G}) = c$ a.s.

(v) If $\mathbb{E}(aX + bY|\mathcal{G}) = a\mathbb{E}(X|\mathcal{G}) + b\mathbb{E}(Y|\mathcal{G})$ a.s., where $a, b \geq 0$ if X and Y are both non-negative, or $a, b \in \mathbb{R}$ if X and Y are both integrable.

(vi) If $X \leq Y$ a.s. then $\mathbb{E}(X|\mathcal{G}) \leq \mathbb{E}(Y|\mathcal{G})$ a.s.

(vii) $|\mathbb{E}(X \mid \mathcal{G})| \leq \mathbb{E}(|X| \mid \mathcal{G})$ a.s.

(viii) If $\{X_n\}_{n=1}^{\infty}$ is an increasing sequence of non-negative random variables converging to X, then

$$\lim_{n\to\infty} \mathbb{E}(X_n|\mathcal{G}) = \mathbb{E}(X|\mathcal{G}) \text{ a.s.}$$

(ix) If $\{X_n\}_{n=1}^{\infty}$ is a sequence of random variables converging a.s. to X and if there is an integrable random variable Y such that $|X_n| \leq Y$ for all n, then

$$\lim_{n\to\infty} \mathbb{E}(X_n|\mathcal{G}) = \mathbb{E}(X|\mathcal{G}) \text{ a.s.}$$

(x) For any $\epsilon > 0$

$$\mathbb{E}(\mathbf{1}_{\{X\geq\epsilon\}}|\mathcal{G}) \leq \frac{\mathbb{E}(X^2|\mathcal{G})}{\epsilon^2}$$

(xi) If $f : \mathbb{R} \to \mathbb{R}$ is a convex function, then

$$f(\mathbb{E}(X|\mathcal{G})) \leq \mathbb{E}(f(X)|\mathcal{G}).$$

(xii) If $p \geq 1$ and $\mathbb{E}|X|^p < \infty$, then

$$|\mathbb{E}(X|\mathcal{G})|^p \leq \mathbb{E}(|X|^p|\mathcal{G}) \text{ a.s.}$$

Taking expectation from both sides of the last inequality and using property (i), gives

$$\mathbb{E}(|\mathbb{E}(X|\mathcal{G})|^p) \leq \mathbb{E}(|X|^p) < \infty.$$

(xiii) If $\mathbb{E}|X|^p < \infty$ and $\mathbb{E}|Y|^q < \infty$, where $p, q \in (1, \infty)$ satisfy $1/p + 1/q = 1$, then

$$|\mathbb{E}(XY|\mathcal{G})| \leq (\mathbb{E}(|X|^p|\mathcal{G})^{1/p}(\mathbb{E}(|Y|^q|\mathcal{G})^{1/q}.$$

(xiv) If $X(\omega) = Y(\omega)$ for all $\omega \in A \in \mathcal{G}$, then $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(Y|\mathcal{G})$ almost surely on A.

(xv) Show that for every $A \in \mathcal{F}$ and $B \in \mathcal{G}$, with $\mathbb{P}(A) > 0$, we have

$$\mathbb{P}(B|A) = \frac{\int_B \mathbb{P}(A|\mathcal{G})\, d\mathbb{P}}{\int_\Omega \mathbb{P}(A|\mathcal{G})\, d\mathbb{P}}.$$

Show that when $\mathcal{G}$ is generated by a partition $B_1, B_2, \ldots, B_n$ of Ω, with $\mathbb{P}(B_i) > 0$ for all i, then this equality reduces to the Bayes's formula

$$\mathbb{P}(B_i|A) = \frac{\mathbb{P}(A|B_i)\mathbb{P}(B_i)}{\sum_{k=1}^n \mathbb{P}(A|B_k)\mathbb{P}(B_k)}.$$

(xvi) Give an example on $\Omega = \{a, b, c\}$ in which

$$\mathbb{E}(\mathbb{E}(X|\mathcal{G}_1)|\mathcal{G}_2) \neq \mathbb{E}(\mathbb{E}(X|\mathcal{G}_2)|\mathcal{G}_1).$$

(xvii) Show that if $\mathbb{E}(Y|\mathcal{G}) = X$ and $\mathbb{E}X^2 = \mathbb{E}Y^2 < \infty$, then $X = Y$ almost surely.