STAT 9657 — Problem Set 2

You may use any results from the course notes when solving the following problems.

(i) (a) Fix two subsets $A, B \subseteq \Omega$ with $A \subseteq B$. What is the σ-algebra generated by $\{A, B\}$?
(b) Fix two subsets $A, B \subseteq \Omega$ with $A \cap B = \emptyset$. What is the σ-algebra generated by $\{A, B\}$?
(c) Fix two subsets $A, B \subseteq \Omega$. What is the σ-algebra generated by $\{A, B\}$?

(ii) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a measure space and let $A_1, A_2, ... \in \mathcal{F}$ be any sets. Show that
(1) $\mathbb{P}\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \lim_{m\to\infty} \mathbb{P}\Big(\bigcup_{n=1}^{m} A_n\Big)$.
(2) If $\mathbb{P}(A_1) < \infty$ then $\mathbb{P}\Big(\bigcap_{n=1}^{\infty} A_n\Big) = \lim_{m\to\infty} \mathbb{P}\Big(\bigcap_{n=1}^{m} A_n\Big)$.

(iii) If $A_1, ..., A_n \in \mathcal{F}$ are such that $\mathbb{P}(A_1) < \infty$ and $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ show that

$$\lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}\Big(\bigcap_{i=1}^{\infty} A_i\Big).$$

(iv) Give an example where the union of two σ-algebras is not a σ-algebra.

(v) Let $x \in \mathbb{R}$. Is it true that $\{x\} \in \mathcal{B}(\mathbb{R})$, where $\{x\}$ is the set consisting of the element x only? Recall that $\mathbb{Q}$ is the set of rational numbers. Is it true that $\mathbb{Q} \in \mathcal{B}(\mathbb{R})$?

(vi) For any sets $B, A_1, A_2, ... \subseteq \Omega$ we have

$$\Big(\bigcup_{i=1}^{\infty} A_i\Big) \cap B = \bigcup_{i=1}^{\infty} (A_i \cap B) \quad \text{and} \quad \Big(\bigcap_{i=1}^{\infty} A_i\Big) \cup B = \bigcap_{i=1}^{\infty} (A_i \cup B).$$

(vii) A collection of sets $\mathcal{F} \subseteq 2^{\Omega}$ is called a **monotonic class** if for any $A_1, A_2, ... \in \mathcal{F}$, such that $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$, we have $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$. Show that if a collection of sets $\mathcal{F} \subseteq 2^{\Omega}$ is an algebra and a monotonic class, then $\mathcal{F}$ is a σ-algebra.

(viii) Suppose $\mathcal{F}$ is a σ-algebra on Ω. Let $\bar{\Omega} \subseteq \Omega$. Show that the collection of sets

$$\bar{\mathcal{F}} := \{\bar{\Omega} \cap A : A \in \mathcal{F}\}$$

is a σ-algebra on $\bar{\Omega}$.

(ix) Show the equality $\sigma(\mathcal{F}_0) = \sigma(\mathcal{F}_2) = \sigma(\mathcal{F}_4)$ in Proposition 36.

(x) Show that $\sigma(\sigma(\mathcal{F})) = \sigma(\mathcal{F})$ for any collection of sets $\mathcal{F} \subseteq 2^{\Omega}$.

(xi) Let $\Omega \neq \emptyset$, $A \subseteq \Omega$, $A \neq \emptyset$, and $\mathcal{F} := 2^{\Omega}$. For any $B \subseteq \Omega$ define

$$\mathbb{P}(B) := \begin{cases} 1 & \text{if } B \cap A \neq \emptyset, \\ 0 & \text{if } B \cap A = \emptyset. \end{cases}$$

Is $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space?

(xii) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $A \in \mathcal{F}$ with $\mathbb{P}(A) > 0$. Show that $(\Omega, \mathcal{F}, \mu)$ is a probability space, where $\mu(B) := \mathbb{P}(B|A)$.

(xiii) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. If $A, B, C \in \mathcal{F}$ are three independent events, show that A^c, B^c, C^c are also independent.

(xiv) Let $\Omega = \{(k,l) : 1 \le k, l \le 6\}$, $\mathcal{F} := 2^\Omega$, and $\mathbb{P}(\{(k,l)\}) = \frac{1}{36}$ be the model of rolling two dice. Define the events

$$\begin{aligned} A &:= \{(k,l) : l = 1, 2 \text{ or } 5\}, \\ B &:= \{(k,l) : l = 4, 5 \text{ or } 6\}, \\ C &:= \{(k,l) : k + l = 9\}. \end{aligned}$$

Are the events A, B, C independent?

(xv) Let $\mathbb{Q}$ be a finite measure on $\mathcal{B}(\mathbb{R})$. Define the function $F(x) := \mathbb{Q}((-\infty, x])$. Show that F is increasing, right-continuous, and $\mathbb{Q}((a,b]) = F(b) - F(a)$ for all $-\infty < a \le b < \infty$.

(xvi) Let $\mathbb{P}$ be the measure in Theorem 56 from the course notes. Show that

(a) $\mathbb{P}((-\infty, b)) = F(b-) - \lim_{x \to -\infty} F(x)$;

(b) $\mathbb{P}((a,b)) = F(b-) - F(a)$;

(c) $\mathbb{P}(\{a\}) = F(a) - F(a-)$;

(d) $\mathbb{P}([a,b)) = F(b-) - F(a-)$;

(e) $\mathbb{P}([a,b]) = F(b) - F(a-)$.

(f) $\mathbb{P}((a,\infty)) = \lim_{x \to \infty} F(x) - F(a)$;

(g) $\mathbb{P}([a,\infty)) = \lim_{x \to \infty} F(x) - F(a-)$;

(h) If, in addition, the function F is continuous, then

i. $\mathbb{P}((-\infty, b]) = \mathbb{P}((-\infty, b)) = F(b) - \lim_{x \to -\infty} F(x)$;

ii. $\mathbb{P}((a,b)) = \mathbb{P}([a,b)) = \mathbb{P}((a,b]) = \mathbb{P}([a,b]) = F(b) - F(a)$;

iii. $\mathbb{P}(\{a\}) = 0$.

(i) If $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$, show that $\mathbb{P}$ is a probability measure.