STAT 9657 — Problem Set 3

You may use any results from the course notes when solving the following problems.

(i) Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be two measurable spaces. Let $\mathcal{G}_1$ be a collection of subsets of Ω_1 that generates $\mathcal{F}_1$, that is $\sigma(\mathcal{G}_1) = \mathcal{F}_1$. Let $\mathcal{G}_2$ be a collection of subsets of Ω_2 that generates $\mathcal{F}_2$, that is $\sigma(\mathcal{G}_2) = \mathcal{F}_2$. Suppose in addition, that $\Omega_1 \in \mathcal{G}_1$ and $\Omega_2 \in \mathcal{G}_2$. Then, the collection of rectangles

$$\{A \times B : A \in \mathcal{G}_1, B \in \mathcal{G}_2\}$$

generates the σ-algebra $\mathcal{F}_1 \otimes \mathcal{F}_2$.

(ii) Define the function $F(x)$ equal to 0 if $x < 0$; to $1/2$ if $0 \leq x < 2$ and to 1 if $2 \leq x$. What is the measure on $\mathcal{B}(\mathbb{R})$ determined by $F(x)$.

(iii) Suppose the sequence $q_1, q_2, ...$ contains all rational numbers, each one exactly once. Let $\alpha_i > 0$ be such that $\sum_{i=1}^{\infty} \alpha_i = 1$. Define the function

$$F(x) = \sum_{i=1}^{\infty} \alpha_i \mathbf{1}_{[q_i,\infty)}(x).$$

Show that $F(x)$ is increasing and right-continuous. What is the measure on $\mathcal{B}(\mathbb{R})$ determined by $F(x)$? (Explain only what the measure is on the sets of the form $(a, b]$, $-\infty < a \leq b < \infty$.)

(iv) Let $A_1, A_2, ..., A_n$ be events, where $n \geq 2$. Show that

$$\begin{aligned}\mathbb{P}(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{1\leq i<j\leq n} \mathbb{P}(A_i \cap A_j) + \sum_{1\leq i<j<k\leq n} \mathbb{P}(A_i \cap A_j \cap A_k) \\ - \cdots + (-1)^{n+1}\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n).\end{aligned}$$

(v) Let $\{A_n\}$ be a sequence of events in a probability space, such that $\mathbb{P}(A_n) = 1$ for all $n = 1, 2, ...$ Show that $\mathbb{P}(\cap_{n=1}^{\infty} A_n) = 1$.

(vi) Ten percent of the surface of a sphere is coloured blue, the rest is red. Show that no matter how the colours are distributed, it is possible to inscribe a cube in the sphere with all its vertices red.

(vii) Let $A_1, A_2, \ldots, A_n$ be n events in Ω. Consider the sequence

$$A_1, A_2, \ldots, A_n, A_1, A_2, \ldots, A_n, A_1, A_2, \ldots, A_n, \ldots$$

What is liminf and limsup of that sequence?

(viii) Let $A_1, A_2, \ldots, A_n$ be n events in Ω. Consider the sequence

$$A_1, A_2, \ldots, A_n, \emptyset, \emptyset, \emptyset, \ldots$$

What is liminf and limsup of that sequence?

We say that the sequence of sets $\{A_n\}$ converge to A if $\limsup A_n = \liminf A_n = A$, in which case we denote the limit by $\lim A_n = A$.

(ix) a) Let $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ be an increasing sequence of events in Ω. What is liminf and limsup of that sequence?
b) Let $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ be a decreasing sequence of events in Ω. What is liminf and limsup of that sequence?

(x) Let $\{A_n\}$ be a sequence of events. Let $C_n := \cap_{i=n}^{\infty} A_i$ and $B_n := \cup_{i=n}^{\infty} A_i$, let $C := \liminf A_n$ and $B := \limsup A_n$. Show that
a) If A_n converges to A then $\mathbb{P}(A_n)$ converges to $\mathbb{P}(A)$.
b) If B_n is independent of C_n for every n, then B is independent of C.
c) If B_n is independent of C_n for every n and if A_n converges to A then $\mathbb{P}(A)$ equals zero or one.

(xi) Give an example of a sequence of sets $\{A_n\}$ that converges to the empty set, but such that the sequence $\{|A_n|\}$ converges to infinity. (Recall that $|A_n|$ is the number of elements in the set A_n.)

(xii) Suppose you have billiard balls, which are numbered $0, 1, 2, \ldots$. At each step of the game you add two balls to a bag and you withdraw one. More precisely, at at one minute before noon balls numbered 0 and 1 are placed in the bag and ball number 0 is removed. At 1/2 minute before noon balls numbered 2 and 3 are added, and ball number 1 is removed. At 1/3 minute before noon balls 4 and 5 are added and ball number 2 is taken out. This process is continued, and the question is asked: How many balls are in the bag at noon?

(xiii) Let $A_1, A_2, \ldots$ be a sequence of events from a probability space. Show that $\mathbb{P}(\limsup A_n) \geq \limsup_{n\to\infty} \mathbb{P}(A_n)$. Is there analogous relationship for liminf?

For the next problems, you may or may not have to use Borel-Cantelli lemma.

(xiv) Let $X_1, X_2, \ldots$ be non-negative, independent and identically distributed random variables with infinite mean. Show that

$$\mathbb{P}\Big(\limsup_{n\to\infty} \frac{X_n}{n} = \infty \Big) = 1.$$

(xv) Let $X_1, X_2, \ldots$ be independent and exponentially distributed with parameter 1. Show that

$$\mathbb{P}\Big(\limsup_{n\to\infty} \frac{X_n}{\log(n)} = 1 \Big) = 1.$$

Hint: consider the events $A_n := \{X_n / \log(n) \geq 1 + \epsilon\}$ and consider two cases when ϵ is positive or negative number close to zero.

(xvi) a) Show that for $x > 0$, the following bounds hold

$$\Big(\frac{1}{x} - \frac{1}{x^3}\Big)e^{-\frac{x^2}{2}} \le \int_x^\infty e^{-\frac{y^2}{2}}\,dy \le \frac{1}{x}e^{-\frac{x^2}{2}}.$$

Hint: For the first inequality show and use that

$$\int_x^\infty \Big(1 - \frac{3}{y^4}\Big)e^{-\frac{y^2}{2}}\,dy = \Big(\frac{1}{x} - \frac{1}{x^3}\Big)e^{-\frac{x^2}{2}}$$

For the second change the variable $y = x + z$.

b) Let $X_1, X_2, ...$ be independent and identically distributed $N(0,1)$ random variables. Show that

$$\mathbb{P}\Big(\limsup_{n\to\infty} \frac{|X_n|}{\sqrt{\log(n)}} = \sqrt{2}\Big) = 1.$$

Hint: consider the events $A_n := \{|X_n| \ge \sqrt{2\log(n)}(1+\epsilon)\}$.

(xvii) Let $\{A_n\}_{n=1}^\infty$ be a sequence of events. Show that if $\lim_{n\to\infty} \mathbb{P}(A_n) \to 0$ and $\sum_{n=1}^\infty \mathbb{P}(A_n^c \cap A_{n+1}) < \infty$, then $\mathbb{P}(\limsup A_n) = 0$.

(xviii) Let $X_1, X_2, ...$ be independent and exponentially distributed with parameter 1. Show that

$$\mathbb{P}\Big(\lim_{n\to\infty} \big(\max_{1\le k\le n} X_k\big)/\log(n) = 1\Big) = 1,$$

by executing the following strategy. Let

$$B_n := \{\big(\max_{1\le k\le n} X_k\big)/\log(n) \ge 1-\epsilon\} \text{ and } C_n := \{\big(\max_{1\le k\le n} X_k\big)/\log(n) \le 1+\epsilon\}.$$

Show that $\mathbb{P}(B_n^c \cup C_n^c) = (1 - 1/n^{1-\epsilon})^n + 1 - (1 - 1/n^{1+\epsilon})^n$. Let $A_n := B_n^c \cup C_n^c$ and show that the sequence $\{A_n\}_{n=1}^\infty$ satisfies the conditions from the previous exercise. To do that, show that $A_n^c \cap A_{n+1} \subseteq \{\max_{i=1,\dots,n+1} X_i \le (1-\epsilon)\log(n+1)\} \cup \{X_{n+1} \ge (1+\epsilon)\log(n+1)\}$ and use it to make the estimate $\mathbb{P}(A_n^c \cap A_{n+1}) \le (1 - 1/(n+1)^{1-\epsilon})^{n+1} + 1/(n+1)^{1+\epsilon}$.

(xix) Consider a sequence of random variables $X_1, X_2, ...$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

We say that the sequence $\{X_n\}$ converges to the random variable X almost surely, if there is a set $A \subset \Omega$ with measure 1 such that $\lim_{n\to\infty} X_n(\omega) = X(\omega)$ for all $\omega \in A$.

We say that the sequence $\{X_n\}$ completely converges to the random variable X if

$$\sum_{n=1}^\infty \mathbb{P}(|X_n - X| > \epsilon) < \infty \text{ for all } \epsilon > 0.$$

Show that if $\{X_n\}$ completely converges to X then it converges to X almost surely.