STAT 9657 — Problem Set 4

You may use any results from the course notes when solving the following problems.

(i) If X and Y are random variables on $(\Omega, \mathcal{F})$, then so are XY, X/Y provided that $Y \neq 0$, and $|X|$.

(ii) Show that step-functions are measurable. Hint: use the fact that by taking all possible intersections of the sets A_i and by adding appropriate complements, we can find a collection of sets $C_1, ..., C_N \in \mathcal{F}$ such that

  (a) $C_j \cap C_k = \emptyset$ for $j \neq k$;

  (b) $\cup_{j=1}^{N} C_j = \cup_{i=1}^{n} A_i$; and

  (c) for every set A_i there is an index set $I_i \subseteq \{1, 2, ..., N\}$ such that $A_i = \cup_{j \in I_i} C_j$.

(iii) Let the function $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a random variable with $X \geq 0$. Show that there is a sequence $\{X_n\}$ of step-functions $X_n : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $0 \leq X_1(\omega) \leq X_2(\omega) \leq \cdots \leq X(\omega)$ and $X(\omega) = \lim_{n\to\infty} X_n(\omega)$ for all $\omega \in \Omega$.

(iv) Let the function $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a random variable. Show that there is a sequence $\{X_n\}$ of step-functions $X_n : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $0 \leq |X_1(\omega)| \leq |X_2(\omega)| \leq \cdots \leq |X(\omega)|$ and $X(\omega) = \lim_{n\to\infty} X_n(\omega)$ for all $\omega \in \Omega$.

(v) Let $X : \Omega \to S$ be a function between the sets Ω and S. Let $\mathcal{S}$ be a σ-algebra on S. Show that

  (a) The collection of subsets $\sigma(X) := \{X^{-1}(A) : A \in \mathcal{S}\}$ is a σ-algebra on Ω. This σ-algebra is called the **σ-algebra generated by** X.

  (b) If $\mathcal{F}$ is a σ-algebra on Ω such that $X : (\Omega, \mathcal{F}) \to (S, \mathcal{S})$ is measurable, then $\sigma(X) \subseteq \mathcal{F}$. Hence $\sigma(X)$ is the smallest σ-algebra such that $X : (\Omega, \sigma(X)) \to (S, \mathcal{S})$ is measurable.

  (c) Let $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a random variable. Let $\mathcal{S}_0 := \{(-\infty, x] : x \in \mathbb{R}\}$. Show that $\sigma(X)$ is generated by $\{X^{-1}(A) : A \in \mathcal{S}_0\}$.

(vi) Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a measure space and let $(S, \mathcal{S})$ be a measurable space. Let $X : (\Omega, \mathcal{F}) \to (S, \mathcal{S})$ be a measurable function. For every set $B \in \mathcal{S}$ define

$$\mathbb{P}_X(B) := \mathbb{P}(\{\omega \in \Omega : X(\omega) \in B\}) = \mathbb{P}(X^{-1}(B)).$$

Show that $\mathbb{P}_X$ is a measure on $\mathcal{S}$. If $\mathbb{P}$ is a probability measure, show that $\mathbb{P}_X$ is also a probability measure. The measure $\mathbb{P}_X$ is called the **image of $\mathbb{P}$ under** X or the **law of** X.

(vii) Let $\mathbb{P}_1$ and $\mathbb{P}_2$ be two measures on $\mathcal{B}(\mathbb{R})$. Show that $\mathbb{P}_1 = \mathbb{P}_2$ if and only if $\mathbb{P}_1((-\infty, x]) = \mathbb{P}_2((-\infty, x])$ for all $x \in \mathbb{R}$.

(viii) a) Let $(\Omega, \mathcal{S})$ be a measurable space and let $\mathcal{M}$ be a collection of functions from Ω to $\mathbb{R}$ containing all step functions. Suppose that if $\{X_n\}$ is a sequence of functions from $\mathcal{M}$ converging to X then X is also in $\mathcal{M}$. Show that $\mathcal{M}$ contains all measurable real-valued functions (i.e. random variables) on $(\Omega, \mathcal{S})$.

(b) Let $X, Y : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be two random variables. There exists a measurable function $f : (\mathbb{R}, \mathcal{B}(\mathbb{R})) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $Y = f(X)$ if and only if $\sigma(Y) \subseteq \sigma(X)$. Hint: let $\mathcal{S} := \sigma(X)$ and consider the set $\mathcal{M}$ of all functions of the form $f(X(\omega))$ for some measurable $f : \mathbb{R} \to \mathbb{R}$.