STAT 9657 — Problem Set 5

You may use any results from the course notes when solving the following problems.

(i) Show that Step 2 in the definition of expected value is "backwards" compatible with Step 1. That is, if X is a non-negative step function, then $\mathbb{E}X = \sup\{\mathbb{E}Z : 0 \leq Z(\omega) \leq X(\omega) \text{ for all } \omega \in \Omega;\ Z \text{ is a step function}\}$.

(ii) Show that there are at most countably many $x \in \mathbb{R}$ where a distribution function F is not continuous.

(iii) Let F be a distribution function. Recall that the quantile function is defined by

$$F^{-1}(\omega) := \sup\{y : F(y) < \omega\} \quad \text{for all } \omega \in (0,1).$$

Show that

(a) If $z < F^{-1}(\omega)$ then $F(z) < \omega$

(b) $F^{-1} \circ F(z) \leq z$ for all $z \in \mathbb{R}$ such that $0 < F(z) < 1$

(c) $F \circ F^{-1}(\omega) \geq \omega$ for all $\omega \in (0,1)$

(iv) Let Y be a random variable uniformly distributed in $(0,1)$. This means that there is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $Y : (\Omega, \mathcal{F}, \mathbb{P}) \to \big((0,1), \mathcal{B}(0,1)\big)$ satisfies $\mathbb{P}(Y \leq a) = a$ for all $a \in (0,1)$. Let F be a distribution function. Recall that F^{-1} can be viewed as a random variable on $\big((0,1), \mathcal{B}(0,1)\big)$ with the Lebesgue measure. Let $X := F^{-1}(Y)$. Note that this is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$. Show that $F_X(x) = F(x)$ for all $x \in \mathbb{R}$.

(v) (a) Let $Y : (\Omega, \mathcal{F}, \mathbb{P}) \to \big(\mathbb{R}, \mathcal{B}(\mathbb{R})\big)$ be a random variable with c.d.f. F_Y. Suppose F_Y is continuous function. Show that $F_Y(Y)$ is uniformly distributed on $(0,1)$.
(b) Conversely, let $Y : (\Omega, \mathcal{F}, \mathbb{P}) \to \big(\mathbb{R}, \mathcal{B}(\mathbb{R})\big)$ be a random variable and let F be a continuous distribution function. If $F(Y)$ is uniformly distributed on $(0,1)$ then $F_Y = F$.

(vi) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $X : \Omega \to \mathbb{R}$ be a random variable. Show that if $X \geq 0$ a.s. and $\mathbb{E}X = 0$ then $X = 0$ a.s.

(vii) Let $X_1, X_2, \ldots$ be positive random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Show that

$$\sum_{k=1}^{\infty} \mathbb{E}X_k = \mathbb{E}\Big(\sum_{k=1}^{\infty} X_k\Big).$$

(viii) Let $X : (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ be a non-negative, integrable random variable with $\mathbb{E}X > 0$. Define the function

$$\mu(A) = \frac{1}{\mathbb{E}X} \int_A X \, d\mathbb{P}$$

for all $A \in \mathcal{F}$.

a) Show that μ is a probability measure on $\mathcal{F}$.

b) Show that for any random variable $Y : (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$, we have

$$\int_\Omega Y \, d\mu = \frac{1}{\mathbb{E}X} \int_\Omega YX \, d\mathbb{P},$$

provided Y is integrable with respect to μ. Hint: show the equality for step-functions, then for non-negative, then for any.