STAT 9657 — Problem Set 6

You may use any results from the course notes when solving the following problems.

(i) If $\mathbb{E}X$ exists and $c \in \mathbb{R}$ then $\mathbb{E}(cX)$ exists and $\mathbb{E}(cX) = c\mathbb{E}X$.

(ii) If $X \le Y$ and $\mathbb{E}|X| < \infty$, then $\mathbb{E}X \le \mathbb{E}Y$.

(iii) If $X = Y$ a.s. and the expectation of one of them exists, then so does the expectation of the other and $\mathbb{E}X = \mathbb{E}Y$.

(iv) If X and Y are integrable then $aX + bY$ is integrable and $\mathbb{E}(aX + bY) = a\mathbb{E}X + b\mathbb{E}Y$ for all $a, b \in \mathbb{R}$.

(v) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $X, X_1, X_2, ... : \Omega \to \mathbb{R}$ be random variables. If $0 \ge X_n \downarrow X$ a.s., then

$$\lim_{n\to\infty} \mathbb{E}X_n = \mathbb{E}X.$$

(vi) If $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function and $X_1, ..., X_n$ are random variables. Show that $f(\mathbb{E}X_1, ..., \mathbb{E}X_n) \le \mathbb{E}f(X_1, ..., X_n)$ provided that $\mathbb{E}|X_i| < \infty$ for all $i = 1, ..., n$. Hint: use the fact that at every $x_0 = (x_{0,1}, ..., x_{0,n}) \in \mathbb{R}^n$ there is a supporting hyperplane to the graph of $f(x)$ at the point $(x_0, f(x_0))$. That means, that there is a vector $a \in \mathbb{R}^n$ and a number $b \in \mathbb{R}$ such that $a_1x_1 + \cdots + a_nx_n + b \le f(x_1, ..., x_n)$ for every vector $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and $a_1x_{0,1} + \cdots + a_nx_{0,n} + b = f(x_{0,1}, ..., x_{0,n})$.

(vii) For $0 < p < q < \infty$ we have that $(\mathbb{E}|X|^p)^{1/p} \le (\mathbb{E}|X|^q)^{1/q}$.

(viii) Let $X \ge 0$ be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ and let $p > 0$, then

$$\mathbb{E}X^p = \int_0^\infty px^{p-1}\mathbb{P}(X > x)\, dx.$$

(ix) a) Suppose $\mathcal{F}_{i,j}$, $1 \le i \le n$, $1 \le j \le m_i$ are independent σ-algebras. Let $\mathcal{G}_i := \sigma(\cup_{j=1}^{m_i}\mathcal{F}_{i,j})$ for $i = 1, ..., n$. Show that $\mathcal{G}_1, ..., \mathcal{G}_n$ are independent σ-algebras.

b) Let $X_1, ..., X_n$ be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{F}_i := \sigma(X_i)$, $i = 1, ..., n$ be the σ-algebras that they generate. Consider the function $(X_1, ..., X_n) : \Omega \to (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Show that the σ-algebra that it generates on Ω is

$$\sigma((X_1, ..., X_n)) = \sigma(\cup_{i=1}^n \mathcal{F}_i).$$

c) Suppose $X_{i,j}$, $1 \le i \le n$, $1 \le j \le m_i$ are independent random variables. Suppose the functions $f_i : \mathbb{R}^{m_i} \to \mathbb{R}$ are measurable, and let $Y_i := f_i(X_{i,1}, ..., X_{i,m_i})$ for $i = 1, ..., , n$. Show that the random variables $Y_1, ..., Y_n$ are also independent.

(x) Let $X > 0$ be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$, show that

$$\lim_{y\to\infty} y\mathbb{E}((1/X)\mathbf{1}_{\{X>y\}}) = 0 \text{ and } \lim_{y\downarrow 0} y\mathbb{E}((1/X)\mathbf{1}_{\{X>y\}}) = 0.$$

(Do not assume that $\mathbb{E}(1/X) < \infty$.)

(xi) If $X = normal(\alpha, a)$ and $Y = normal(\beta, b)$ are independent, show that $X+Y = normal(\alpha+\beta, a+b)$. (Here, a and b are variances.)

(xii) If $X = gamma(\alpha, \lambda)$ and $Y = gamma(\beta, \lambda)$ are independent, show that $X+Y = gamma(\alpha+\beta, \lambda)$.