STAT 9657 — Problem Set 7

You may use any results from the course notes when solving the following problems.

(i) Show that two random variables X and Y are independent if and only if the σ-algebras $\sigma(X)$ and $\sigma(Y)$ are independent. The same result holds true for any number of random variables. Hint: note that $\sigma(X)$ contains Ω.

(ii) Let $X_1, ..., X_n$ be random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. If the c.d.f. $F(x_1, ..., x_n)$ has **density** $f(x_1, ..., x_n)$ that is

$$F(x_1, ..., x_n) = \int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_1} f(y_1, ..., y_n)\, dy_1 ... dy_n$$

and if $f(x_1, ..., x_n) = g_1(x_1) \cdots g_n(x_n)$, where $g_i \geq 0$ is a measurable function $i = 1, ..., n$, then $X_1, ..., X_n$ are independent.

(iii) Let the joint distribution of X and Y be as follows

		Y		
		1	0	-1
	1	0	a	0
X	0	b	c	b
	-1	0	a	0

where the numbers a, b, c are strictly positive with $2a + 2b + c = 1$. Show that X and Y are not independent but we have $\mathbb{E}(XY) = \mathbb{E}X\mathbb{E}Y$.

(iv) Let X be a random variable and let $c \in \mathbb{R}$. Show that

$$\int_{\mathbb{R}} \big(F_{X-c}(x) - F_X(x)\big)\, dx = c.$$

(v) If X and Y are independent integer-valued random variables then

$$\mathbb{P}(X + Y = n) = \sum_{m-\text{integer}} \mathbb{P}(X = m)\mathbb{P}(Y = n - m).$$

(vi) Let $X = \mathit{Poisson}(\lambda)$ and $Y = \mathit{Poisson}(\mu)$ are independent random variables. Show that $X + Y = \mathit{Poisson}(\lambda + \mu)$.

(vii) Let $X = \mathit{Binomial}(n, p)$ and $Y = \mathit{Binomial}(m, p)$ are independent random variables. Show that $X + Y = \mathit{Binomial}(n + m, p)$.

(viii) Show that the sum of n independent $\mathit{Bernoulli}(p)$ random variables is $\mathit{Binomial}(n, p)$.

(ix) Let X_1,...,X_n be independent *exponential(λ)* random variables. Show that $X_1 + \cdots + X_n$ is $\mathit{gamma}(n, \lambda)$.

(x) Let $X, Y \geq 0$ be independent with c.d.f.'s F_X and F_Y. Find the distribution function of XY.

(xi) Suppose that $X_n \xrightarrow{L_p} X$ and $Y_n \xrightarrow{L_q} Y$, where $1/p + 1/q = 1$. Show that $X_n Y_n \xrightarrow{L_1} XY$.

(xii) Let $1 \leq q \leq p < \infty$ and suppose that $X_n \xrightarrow{L_p} X$. Show that $X_n \xrightarrow{L_q} X$.

(xiii) Let $X : (\Omega, \mathcal{F}, \mathbb{P}) \to (S, \mathcal{S})$ be a measurable map. Let $g : (S, \mathcal{S}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a measurable real-valued function. Let $A \in \mathcal{F}$.

(a) Use the change of variable formula to represent the integral

$$\int_A g(X(\omega))\, d\mathbb{P}(\omega)$$

as an integral over S. (Assuming $g(X)\mathbf{1}_A \geqslant 0$ a.s. or $\mathbb{E}|g(X)\mathbf{1}_A| < \infty$.)

(b) Express the measure, over S that you used in part a), in terms of $\mathbb{P}$, A and X only.

(xiv) Show that the following conditions are equivalent

(a) $X_n \xrightarrow{a.s.} X$;

(b) $\lim_{n\to\infty} \mathbb{P}(\sup_{m\geq n} |X_m - X| > \epsilon) = 0$ for all $\epsilon > 0$;

(c) $\mathbb{P}(\text{limsup}\, \{|X_n - X| > \epsilon\}) = 0$ for all $\epsilon > 0$.

(xv) Show that for a random variable Y, we have

$$\mathbb{E}[Y] = \int_0^\infty P(Y > y)\, dy - \int_0^\infty P(Y < -y)\, dy.$$