STAT 9657 — Problem Set 8

You may use any results from the course notes when solving the following problems.

(i) Let $\{X_n\}$ be a sequence of random variables. Show that $\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}$ and $\{\omega \in \Omega : |X_n(\omega) - X(\omega)| > \epsilon\}$ are measurable sets.

(ii) (Part of this exercise was already in Problem Set 5) Let F be a distribution function. Recall that the quantile function is defined by $F^{-1}(\omega) := \sup\{x : F(x) < \omega\}$ for all $\omega \in (0,1)$. Show that

- (a) F^{-1} is increasing, left-continuous function on $(0,1)$
- (b) If $x > F^{-1}(\omega)$ then $F(x) \geq \omega$
- (c) If F^{-1} is continuous at ω, then $x > F^{-1}(\omega)$ implies that $F(x) > \omega$.
- (d) If $F(x) > \omega$ then $x \geq F^{-1}(\omega)$
- (e) If F is continuous at x, then $F(x) > \omega$ implies that $x > F^{-1}(\omega)$
- (f) If $x < F^{-1}(\omega)$ then $F(x) < \omega$
- (g) If $F(x) < \omega$ then $x < F^{-1}(\omega)$
- (h) $F^{-1} \circ F(x) \leq x$ for all $x \in \mathbb{R}$ such that $0 < F(x) < 1$ (what goes wrong when $F(x)$ is 0 or 1?)
- (i) $F \circ F^{-1}(\omega) \geq \omega$ for all $\omega \in (0,1)$
- (j) $F \circ F^{-1} \circ F(x) = F(x)$ for all $x \in \mathbb{R}$ such that $0 < F(x) < 1$
- (k) $F^{-1} \circ F \circ F^{-1}(\omega) = F^{-1}(\omega)$ for all $\omega \in (0,1)$. Consider the case, when $F \circ F^{-1}(\omega) = 1$ separately.
- (l) If F has a jump at x then F^{-1} is constant on $(F(x-), F(x)]$
- (m) If F^{-1} has a jump at ω then F is constant on $[F^{-1}(\omega), F^{-1}(\omega+))$
- (n) The right limit of F^{-1} at ω is $F^{-1}(\omega+) = \inf\{t : F(t) > \omega\}$
- (o) $Y(\omega) := F^{-1}(\omega)$ is a random variable on $\big((0,1), \mathcal{B}(0,1)\big)$ with the Lebesgue measure. Show that $F_Y = F$.

(iii) For any random variables X and Y, any $x \in \mathbb{R}$ and $\epsilon > 0$ we have

$$\{Y \leq x\} \subseteq \{X \leq x + \epsilon\} \cup \{|Y - X| > \epsilon\},$$
$$\{X \leq x - \epsilon\} \subseteq \{Y \leq x\} \cup \{|Y - X| > \epsilon\}.$$

(iv) If $X_n \xrightarrow{d} X$, where $X(\omega) = c$ for every $\omega \in \Omega$, then $X_n \xrightarrow{\mathbb{P}} X$.

(v) Suppose $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$, where $Y(\omega) = c$ for every $\omega \in \Omega$. Show that

- (a) $X_n + Y_n \xrightarrow{d} X + Y$;

(b) If $Z_n - X_n \xrightarrow{d} 0$ then $Z_n \xrightarrow{d} X$;

(c) $X_n Y_n \xrightarrow{d} XY$. (To avoid complications, assume that $Y_n \geq 0$ and $c > 0$.)

(vi) If the events $\{A_n\}_{n=1}^{\infty}$ are independent and $A_n \to A$ then $\mathbb{P}(A) = 0$ or 1. Recall that $A_n \to A$ if $\text{liminf} A_n = \text{limsup} A_n = A$.

(vii) If the random variables $\{X_n\}_{n=1}^{\infty}$ are independent and $X_n \xrightarrow{a.s.} 0$, then $\sum_{n=1}^{\infty} \mathbb{P}(|X_n| \geq c) < \infty$ for any $c > 0$.

(viii) Show that for any random variable X we have

$$\sum_{n=1}^{\infty} \mathbb{P}(|X| \geq n) \leq \mathbb{E}|X| \leq 1 + \sum_{n=1}^{\infty} \mathbb{P}(|X| \geq n).$$

(ix) Let $X_1, X_2, \ldots$ be a sequence of i.i.d., non-negative random variables with infinite mean. Show that

$$\mathbb{P}\Big(\limsup_{n\to\infty} \frac{X_n}{n} = \infty \Big) = 1.$$

(x) Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, with $\mathbb{E}X_k = 0$ and $\operatorname{Var} X_k < \infty$, for all $k = 1, 2, \ldots$ Show that

$$\frac{S_n}{n^p} \xrightarrow{\text{a.s.}} 0$$

for all $p > 1/2$.

(xi) Let $X_1, X_2, \ldots$ be a sequence of independent, integrable, random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, with $\mathbb{E}X_k = 0$ for all $k = 1, 2, \ldots$ Show that

$$\text{If } \frac{S_n}{n} \xrightarrow{\mathbb{P}} 0 \text{ then } \frac{X_n}{n} \xrightarrow{\mathbb{P}} 0$$

(xii) Let $X_1, X_2, \ldots$ be a sequence of independent, integrable, random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, with $\mathbb{E}X_k = 0$ for all $k = 1, 2, \ldots$ Show that

$$\text{If } \frac{S_n}{n} \xrightarrow{\text{a.s.}} 0 \text{ then } \sum_{n=1}^{\infty} \mathbb{P}\{|X_n| \geq \epsilon n\} < \infty \quad \text{for all } \epsilon > 0.$$

(xiii) Prove or disprove the following statement: If $\{X_n\}$ is a sequence of random variables such that $X_n \xrightarrow{a.s.} X$, then $F_{X_n} \xrightarrow{w} F_X$.

(xiv) If $X_n \xrightarrow{d} X$ and $X_n \geq 0$, then $\mathbb{E}X \leq \text{liminf}_{n\to\infty} \mathbb{E}X_n$.

(xv) Let $X, Y_1, Y_2, \ldots$ be random variables with $\mathbb{P}(Y_n = 1) = 1/n$ and $\mathbb{P}(Y_n = 0) = 1 - 1/n$. Let $Z_n = X + Y_n$. Prove that $Z_n \xrightarrow{d} X$.