STAT 9657 — Problem Set 9

You may use any results from the course notes when solving the following problems.

(i) Calculate the following characteristic functions.

(a) A binomial random variable X with $\mathbb{P}(X=1)=\mathbb{P}(X=-1)=1/2$.
(b) Poisson random variable X with $\mathbb{P}(X=k)=e^{-\lambda}\lambda^k/k!$.
(c) Normal random variable X with density $(2\pi\sigma^2)^{-1/2}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$.
(d) Uniform random variable X on (a,b).
(e) Exponential random variable X with density $\lambda e^{-\lambda x}$.

(ii) Show that if $X_1,\dots,X_n$ are independent and uniformly distributes on $(-1,1)$, then for $n\geq 2$, $X_1+\cdots+X_n$ has density

$$f(x)=\frac{1}{\pi}\int_0^\infty (\sin(t)/t)^n\cos(tx)\,dt.$$

(iii) Let $X_1,X_2,\dots$ be independent and let $S_n:=X_1+\cdots+X_n$. Let $\phi_k(t)$ be the characteristic function of X_k. Show that if $S_n\xrightarrow{\text{a.s.}}S_\infty$, then the characteristic function of S_∞ is $\prod_{k=1}^\infty\phi_k(t)$.

(iv) Let $X_1,X_2,\dots$ be independent and identically distributed and let $S_n:=X_1+\cdots+X_n$. Let $\phi(t)$ be the characteristic function of X_k. Show that if $\phi'(0)=ia$ then $S_n/n\xrightarrow{\mathbb{P}}a$.

(v) Let $(\Omega,\mathcal{F})$ be a measurable space and let $\mathbb{P}_1$ and $\mathbb{P}_2$ be probability measures on $\mathcal{F}$.

(a) Show that $\mathbb{P}:=\frac{1}{2}\mathbb{P}_1+\frac{1}{2}\mathbb{P}_2$ is a probability measure on $\mathcal{F}$.
(b) Let $X:(\Omega,\mathcal{F})\to(\mathbb{R},\mathcal{B}(\mathbb{R}))$ be a measurable function integrable with respect to $\mathbb{P}$. Show that $\int_\Omega X\,d\mathbb{P}=\frac{1}{2}\int_\Omega X\,d\mathbb{P}_1+\frac{1}{2}\int_\Omega X\,d\mathbb{P}_2$.
(c) Suppose $F_1,\dots,F_n$ are distribution functions with characteristic functions $\phi_1,\dots,\phi_n$. Let the numbers $\lambda_1,\dots,\lambda_n$ be in $[0,1]$ with sum $\sum_{k=1}^n\lambda_k=1$. Show that $\sum_{k=1}^n\lambda_kF_k$ is a distribution function with characteristic function $\sum_{k=1}^n\lambda_k\phi_k$.

(vi) Let $X_1,X_2,\dots$ be independent and let $S_n:=X_1+\cdots+X_n$. Suppose

$$\mathbb{P}(X_m=m)=\mathbb{P}(X_m=-m)=m^{-2}/2,\qquad \text{for } m\geq 2;\ \text{and}$$
$$\mathbb{P}(X_m=1)=\mathbb{P}(X_m=-1)=(1-m^{-2})/2.$$

Show that $(\operatorname{Var}S_n)/n\to 2$ and that $S_n/\sqrt{n}\xrightarrow{d}N(0,1)$.

(vii) If the random variables $f,g:(\Omega,\mathcal{F},\mathbb{P})\to(\mathbb{R},\mathcal{B}(\mathbb{R}))$ are integrable and satisfy

$$\int_A f\,d\mathbb{P}=\int_A g\,d\mathbb{P}\qquad\text{for all } A\in\mathcal{F}.$$

then $f=g$ a.s.

(viii) Show that the Glivenko-Cantelli theorems follows from the Dvoretzky-Kiefer-Wolfowitz inequality.