Statistics 3858 : Likelihood Ratio for Exponential Distribution

In these two example the rejection rejection region is of the form

$$\{\mathbf{x} : -2\log(\Lambda(\mathbf{x})) > c\}$$

for an appropriate constant c. For a size α test, using Theorem 9.5A we obtain this critical value from a $\chi^2_{(1)}$ distribution. For $\alpha = .05$ we obtain $c = 3.84$. On the surface these appear to be the same, but the set of $\mathbf{x}$ in this rejection region is different for the one and two sided alternatives.

1 One Sided Alternative

$X_i, i = 1, 2, \ldots, n$ iid exponential, λ.

Consider $H_0 : \lambda = \lambda_0$ versus the alternative $\lambda < \lambda_0$. Find the generalized likelihood ratio test and show that it is equivalent to $\bar{X} > c$, in the sense that the rejection region is of the form $\bar{X} > c$.

The null hypothesis is $H_0 : \lambda \in \Theta_0 = \{\lambda_0\}$ and the alternative is $H_A : \lambda \in \Theta_A = \{\lambda : \lambda < \lambda_0\} = (0, \lambda_0)$.

The likelihood function is

$$L(\lambda) = \lambda^n e^{-n\lambda\bar{X}}$$

The generalized likelihood ratio is

$$\Lambda = \frac{\max_{\lambda\in\Theta_0} L(\lambda)}{\max_{\lambda\in\Theta_0\cup\Theta_A} L(\lambda)} \tag{1}$$

The numerator is

$$\lambda_0^n e^{-n\lambda_0\bar{X}}$$

For the denominator we need to find argmax. Solving

$$\frac{\partial \log(L(\lambda))}{\partial\lambda} = \frac{n}{\lambda} - n\bar{X} = 0$$

The solution is $\tilde{\lambda} = \frac{1}{\bar{X}}$. Thus for the denominator the argmax is given by

$$\tilde{\lambda} = \begin{cases} \frac{1}{\bar{X}} & \text{if } \frac{1}{\bar{X}} < \lambda_0 \\ \lambda_0 & \text{otherwise} \end{cases}$$

The likelihood ratio is then given by

$$\Lambda = \begin{cases} \left(\lambda_0 \bar{X}\right)^n e^{-n\lambda_0 \bar{X} + n\frac{1}{\bar{X}}\bar{X}} & \text{if } \frac{1}{\bar{X}} < \lambda_0 \\ \left(\frac{\lambda_0}{\lambda_0}\right)^n e^{-n\lambda_0 \bar{X} + n\lambda_0 \bar{X}} & \text{if } \frac{1}{\bar{X}} \geq \lambda_0 \end{cases}$$

After simplification this is

$$\Lambda = \begin{cases} \left(\lambda_0 \bar{X}\right)^n e^{-n\lambda_0 \bar{X} + n} & \text{if } \frac{1}{\bar{X}} < \lambda_0 \\ 1 & \text{if } \frac{1}{\bar{X}} \geq \lambda_0 \end{cases}$$

Consider the function $g : (0, \infty) \mapsto \mathcal{R}$ given by

$$g(y) = \lambda_0 y e^{-\lambda_0 y + 1}$$

Aside : This function comes from noticing that for $\frac{1}{\bar{X}} < \lambda_0$

$$\Lambda^{1/n} = g(\bar{X})$$

By calculus the student should verify that g is monotonically increasing for $y < \frac{1}{\lambda_0}$ and monotonically decreasing for $y > \frac{1}{\lambda_0}$. Also show

$$\Lambda = \begin{cases} g(\bar{X})^n & \text{if } \frac{1}{\bar{X}} < \lambda_0 \\ 1 & \text{if } \frac{1}{\bar{X}} \geq \lambda_0 \end{cases}$$

The rejection region for the GLR test is

$$R = \{\underline{x} = (x_1, x_2, \ldots, x_n) : \Lambda < c\}$$

where $c < 1$.

In the lecture we fill in the step to show this is the same set as

$$\{\underline{x} : \bar{x} > c^*\}$$

where c^* solves in terms of $y > \frac{1}{\lambda_0}$

$$g(y)^n = c\ .$$

Notice this is in fact sensible since it says that we reject if $\bar{X}_n > \frac{1}{\lambda_0}$, and the alternative is values of λ such that $\frac{1}{\lambda} > \frac{1}{\lambda_0}$.

2 Two Sided Alternative

This example is as above but with $H_0 : \lambda = \lambda_0$ versus $H_A : \lambda \neq \lambda_0$. After appropriate calculations the student should show that the generalized likelihood ratio is

$$\Lambda(X) = \left(\lambda_0 \bar{X}\right)^n e^{-n\lambda_0 \bar{X}+n} \tag{2}$$

The student should sketch the curve of g (same function as in section 1). The rejection region corresponds to the set of $\mathbf{x}$ satisfying

$$\bar{x} < y_L^* \ , \quad \text{or} \quad \bar{x} > y_U^*$$

where the numbers y_L^*, y_U^* are solutions of

$$g(y) = c$$

Notice this is a different set than the rejection region for the one sided alternative.

In class we have Theorem 9.5.A that states, under the null hypothesis, $-2\log(\Lambda)$ converges in distribution to $\chi^2_{(df)}$ where in this case the degrees of freedom is $df = 1 - 0 = 1$. In the rest of this section for this example we verify the limit distribution directly by using a Taylor's approximation of order 2 and an extension of the delta method discussed earlier.

In studying this part we can recognize that $\log(\Lambda(X))$ is a function of $\bar{X}$, and so we will need to study the role of this function and what results from it. As an aside notice that we end up with a function of $\bar{X}$ since it is a sufficient statistic for this statistical model. Notice that $\log(\Lambda(X)) = ng(\bar{X})$ where

$$g(y) = \log(\lambda_0 y) - (\lambda_0 y - 1)$$

This is a different function g than used in the previous part.

Theorem 9.5A is beyond what we can study in our course. However we can get some understanding of this theorem by studying the special case in our problem. It involves a Taylor's expansion and some properties of convergence in distribution.

Expand g about $\mathrm{E}_0(X) = \frac{1}{\lambda_0}$, the mean under the assumption that the null hypothesis holds.

For this function we have

$$g'(y) = \frac{1}{y} - \lambda_0$$

and

$$g''(y) = -\frac{1}{y^2}$$

so that $g'(\frac{1}{\lambda_0}) = 0$ and $g''(\frac{1}{\lambda_0}) = -\lambda_0^2$.

The first order Taylor approximation is

$$g_1(y) = g(1/\lambda_0) + g'(1/\lambda_0)\left(y - \frac{1}{\lambda_0}\right) = 0$$

and the second order approximation is

$$g_2(y) = g_1(y) + \frac{1}{2}g''(\lambda_0)\left(y - \frac{1}{\lambda_0}\right)^2 = -\frac{1}{2}\lambda_0^2\left(y - \frac{1}{\lambda_0}\right)^2$$

The first order Taylor approximation is not very useful so we use the next order approximation.

Thus, noting that $\hat{\lambda} = \bar{X}$

$$\begin{aligned} -2\log(\Lambda(X)) &= -2ng(\bar{X}) \\ &\approx -2ng_2(\bar{X}) \\ &= 2n * \frac{1}{2}\lambda_0^2\left(\bar{X} - \frac{1}{\lambda_0}\right)^2 \\ &= \left(\frac{\sqrt{n}\left(\bar{X} - \frac{1}{\lambda_0}\right)}{\frac{1}{\lambda_0}}\right)^2 \end{aligned}$$

Under the assumption that the null hypothesis is true, the term in the last expression has then approximate distribution of Z^2, where $Z \sim N(0,1)$, that is $\chi^2_{(1)}$.

How would we use this in practice?

$H_0 : \lambda = 2$ versus $H_A : \lambda \neq 2$.

The GLR is

$$\Lambda(X) = \left(\lambda_0 \bar{X}\right)^n e^{-n\lambda_0 \bar{X} + n}$$

Under H_0 this has a $\chi^2_{(1)}$ distribution. To perform this hypothesis test at level $\alpha = .01$ we would use the upper .01 quantile as the critical value, that is $c = 6.63$. Thus the rejection region is

$$R = \{\mathbf{x} : -2\log(\Lambda(x)) > 6.63\}$$