Statistics 9657 : Some Useful Inequalities in Probability

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A very useful inequality is the following :
Let $p \geq 1$. For $x, y \in R$

$$|(x+y)^p| \leq 2^{p-1}\left(|x|^p + |y|^p\right) \ . \tag{1}$$

For $p = 1$ inequality (1) is trivial. Thus it suffices to show it is true for $p > 1$.

Lemma 1 *Suppose that $p > 1$. Let*

$$h(t) = C(1+t^p) - (1+t)^p$$

be a function from $[1, \infty) \mapsto R$. For $C = 2^{p-1}$, $h(t)$ is an increasing function on $[1, \infty)$, and $h(1) = 0$.

Proof of Lemma 1 : $h(1) = 2C - 2^p = 2(C - 2^{p-1}) = 0$.

$$\begin{aligned} h'(t) &= Cpt^{p-1} - p(1+t)^{p-1} \\ &= \frac{p}{(1+t)^{p-1}}\left(C\left(\frac{t}{1+t}\right)^{p-1} - 1\right) \\ &\geq \frac{p}{(1+t)^{p-1}}\left(C\frac{1}{2^{p-1}} - 1\right) \quad \text{for } t \geq 1 \end{aligned}$$

and hence $h'(t) \geq 0$ on $1 \leq t < \infty$.

Proof of Inequality (1) :

Noting that

$$\begin{aligned} |(x+y)^p| &\leq (|x+y|)^p \\ &\leq (|x|+|y|)^p \end{aligned}$$

and thus it is sufficient to prove (1) holds for $x, y \geq 0$.

If either x or y equals 0, the inequality is trivial.

Thus we need only consider the case $x, y > 0$. Without loss of generality suppose that $x \leq y$. Then

$$\begin{aligned} (x+y)^p &= x^p \left(1 + \frac{y}{x}\right)^p \\ &\leq x^p 2^{p-1} \left(1 + \left(\frac{y}{x}\right)^p\right) \\ &= 2^{p-1}\left(x^p + y^p\right) \end{aligned}$$

where the inequality holds by Lemma 1, since $y \geq x > 0$.

Another important inequality is Hölder's Inequality. The proof given here is taken from Royden [1].

Lemma 2 *Let α and β be nonnegative real numbers and suppose that $0 < \lambda < 1$. Then*

$$\alpha^\lambda \beta^{1-\lambda} \leq \lambda\alpha + (1-\lambda)\beta \ .$$

Proof of Lemma 2 :

Let $\phi(t) = (1-\lambda) + \lambda t - t^\lambda$, $t > 0$. The student should verify that, since $\lambda - 1 < 0$, we have $\phi'(t) < 0$ for $t < 1$, $\phi'(1) = 0$ and $\phi'(t) > 0$ for $t > 1$.

Thus for $t \neq 1$, $\phi(t) > \phi(1) = 0$. Thus

$$(1-\lambda) + \lambda t \geq t^\lambda$$

and equality only if $t = 1$.

For $\alpha, \beta > 0$,

$$(1-\lambda) + \lambda\left(\frac{\alpha}{\beta}\right) \geq \left(\frac{\alpha}{\beta}\right)^\lambda \ .$$

After multiplying by β, the inequality follows. If either α or β equals 0, the inequality is trivial.

Let a be a sequence such that

$$||a||_p = \left(\sum_{i=1}^{\infty} |a_i|^p\right)^{\frac{1}{p}} \tag{2}$$

is finite. $||\cdot||_p$ is called the p-norm of the sequence. If $p = \infty$, then

$$||a||_p = ||a||_\infty = \sup_i |a_i||$$

and it is also called the sup-norm.

Theorem 1 (Hölder's Inequality for Sequences) *Suppose p and q are extended real numbers greater than or equal to 1 and $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that a and b are sequences such that $||a||_p < \infty$ and $||b||_q < \infty$. Then*

$$\sum_i |a_i b_i| \leq ||a||_p \cdot ||q||_q \ .$$

Remark : If $p = 1$ then $q = \infty$.

Proof :

Suppose that $p > 1$.

Suppose that $||a||_p = ||q||_q = 1$. Apply Lemma 2 to $\alpha = |a_i|^p$, $\beta = |b_i|^q$ and take $\lambda = \frac{1}{p}$, so that $1 - \lambda = \frac{1}{q}$. Then

$$\begin{aligned} \sum_i |a_i b_i| &= \sum_i |a_i|^{p\frac{1}{p}} \cdot |b_i|^{q\frac{1}{q}} \\ &= \sum_i |a_i|^{p\lambda} \cdot |b_i|^{q(1-\lambda)} \\ &\leq \sum_i \{\lambda |a_i|^p + (1-\lambda)|b_i|^q\} \\ &= \lambda + (1-\lambda) = 1 \ . \end{aligned}$$

Suppose that $||a||_p > 0$ and $||q||_q > 0$. Since $\tilde{a} = \{a_i/||a||_p\}$ and $\tilde{b} = \{b_i/||b||_q\}$ are sequences with p-norm and q-norm equal to 1, the inequality follows again.

If either sequence is 0, the inequality would be $0 \leq 0$, which is indeed true.

Another useful inequality is Minkowski's Inequality. Again we only prove it for sequences.

Theorem 2 (Mikowski's Inequality) *Suppose that a and b have finite p-norms for $1 \leq p \leq \infty$. Then so does $a+b$ and*

$$||a+b||_p \leq ||a||_p + ||b||_p \ .$$

Proof : For $p = 1$ or ∞, the student should verify this at home.

Now suppose $1 < p < \infty$.

By Lemma 1, $|a_i + b_i|^p \leq 2^{p-1}(|a_i|^p + |b_i|^p)$, and the first conclusion in the Theorem will now follow.

Note

$$|a_i + b_i|^p \leq |a_i + b_i|^{p-1}(|a_i| + |b_i|) \leq |a_i + b_i|^{p-1}|a_i| + |a_i + b_i|^{p-1}|b_i| \ .$$

Also $q(p-1) = p$.

Then

$$\begin{aligned}
\sum_i |a_i + b_i|^p &\leq \sum_i |a_i + b_i|^{p-1}|a_i| + \sum_i |a_i + b_i|^{p-1}|b_i| \\
&\leq \left(\sum_i |a_i + b_i|^{(p-1)q}\right)^{\frac{1}{q}} ||a||_p + \left(\sum_i |a_i + b_i|^{(p-1)q}\right)^{\frac{1}{q}} ||b||_p \\
&\leq \left(\sum_i |a_i + b_i|^{p}\right)^{\frac{1}{q}} ||a||_p + \left(\sum_i |a_i + b_i|^{p}\right)^{\frac{1}{q}} ||b||_p \\
&= \left(\sum_i |a_i + b_i|^{p}\right)^{\frac{1}{q}} (||a||_p + ||b||_p) \ .
\end{aligned}$$

The second inequality follows from Hölder's Inequality applied to each of the two pieces.

Comparing the first line to the last, after dividing by the first term on the last line, we have

$$\left(\sum_i |a_i + b_i|^p\right)^{1-\frac{1}{q}} \leq (||a||_p + ||b||_p)$$

Noting that $1 - \frac{1}{q} = \frac{1}{p}$ we thus have

$$\left(\sum_i |a_i + b_i|^p\right)^{\frac{1}{p}} \leq (||a||_p + ||b||_p)$$

or equivalently

$$||a+b||_p \leq ||a||_p + ||b||_p \ .$$

References

[1] Royden, H. L. (1968). *Real Analysis*. MacMillan.