PART V

LONG MEMORY MODELLING

The **Hurst phenomenon** created one of the most interesting, controversial and long-lasting scientific debates ever to arise in the field of hydrology. The genesis of the Hurst phenomenon took place over forty years ago in Egypt. Just after World War II, a British scientist by the name of **Harold Edwin Hurst** became deeply involved in studying how the Nile River could be optimally controlled and utilized for the benefit of both Egypt and Sudan. As Director-General of the Physical Department in the Ministry of Public Works in Cairo, Egypt, Hurst was particularly interested in the **long-term storage** requirements of the Nile River. In addition to annual riverflow series, Hurst analyzed a wide variety of other yearly geophysical time series in order to examine the statistical properties of some specific statistics that are closely related to long term storage. These statistical studies led Hurst to develop an empirical law upon which the definition of the Hurst phenomenon is based.

The fact that the Hurst phenomenon arose from scientific work carried out in Egypt provided the controversy with an aura of mystery and intrigue. Was the Hurst phenomenon more difficult to solve than the riddle of the Sphinx? Indeed, a range of explanations has been put forward for solving the Hurst phenomenon. Furthermore, in the process of studying the Hurst phenomenon, many original contributions have been made to the fields of hydrology and statistics.

In Chapter 10, the **Hurst phenomenon** is defined and both theoretical and empirical work related to this phenomenon are described. One spinoff from research connected to Hurst's work is the development of a stochastic model called **fractional Gaussian noise (FGN)**. This model possesses **long memory** (see Section 2.5.3) and was designed for furnishing an explanation to the Hurst phenomenon. As demonstrated in Chapter 10, this long memory model fails to solve the Hurst riddle. Nevertheless, the introduction of FGN into the field of hydrology initiated major theoretical and practical developments in long memory modelling by not only hydrologists but also by statisticians and economists. Probably the most flexible and comprehensive type of long memory model is the **fractional autoregressive-moving average or FARMA model** presented in Chapter 11. In fact, the FARMA family of models is a direct extension of the ARMA class of models defined in Chapter 3.

If FGN modelling cannot provide a reasonable solution to the Hurst phenomenon, then wherein lies the answer? The solution to Hurst's riddle is put forward in Section 10.6 of the next chapter. Simulation experiments demonstrate that when the most appropriate ARMA models are fitted to a wide variety of annual natural time series, a statistic called **the Hurst coefficient is "statistically preserved" by the calibrated ARMA models.** Therefore, although the Hurst coefficient and other related statistics are not directly incorporated as model parameters in the design of an ARMA model, these statistics can still be indirectly accounted for or modelled by ARMA models.

CHAPTER 10

THE HURST PHENOMENON

AND

FRACTIONAL GAUSSIAN NOISE

10.1 INTRODUCTION

Since the original empirical studies of Hurst (1951), the *Hurst phenomenon* has caused extensive research with accompanying academic controversies right up to the present time. The objectives of this chapter are to review and appraise research related to Hurst's work and demonstrate how the Hurst phenomenon can be explained. The views presented in this chapter, as well as research by Hipel (1975), McLeod and Hipel (1978a) and Hipel and McLeod (1978a, 1978b), constitute a fresh approach to the study of the Hurst phenomenon and the related problem of the *preservation of historical statistics* by stochastic models.

In Section 10.2, some statistics related to long term storage requirements of a reservoir are defined using the idea of a *cumulated range*. Subsequently, the various types of Hurst coefficients that have been developed for use in formulae involving the *rescaled adjusted range (RAR)* are given and compared in Section 10.3.1. Because of the flexible statistical properties of the RAR, it is suggested that this is the Hurst statistic of primary concern in water resource applications related to storage.

The roles of both identically independently distributed (IID) variables and correlated random variables for explaining problems related to the Hurst phenomenon are thoroughly investigated in Sections 10.3.2 and 10.3.3, respectively. Simulation studies are used to demonstrate that the RAR is nearly independent of the type of underlying distribution of the random variables and is also a function of the sample size. Of particular importance for correlated processes are stochastic models that can be easily fitted to natural time series and at the same time retain relevant historical statistical characteristics of the data such as the RAR and other related statistics. The ARMA models of Chapter 3 constitute one family of stochastic or time series models which possess the potential for continued extensive utilization in hydrology. The *fractional Gaussian noise (FGN) model* of Section 10.4 is a process that was developed mainly within the hydrological literature (Mandelbrot and Wallis, 1968, 1969a to e) as a means for possibly accounting for the Hurst phenomenon. Although some of the inherent drawbacks of this model are discussed, significant contributions are formulated toward the further statistical maturity of the FGN model in Section 10.4.

As explained in Part III, when any type of time series model is being fitted to a given time series, it is recommended to follow the identification, estimation and diagnostic check stages of model construction. Within Section 10.4, useful model building techniques are presented for allowing FGN models to be applied properly to data sets. More specifically, in Section 10.4.3 an efficient maximum likelihood estimation (MLE) procedure is derived for use at the estimation stage. Simulation studies reveal that the MLE approach is superior to a previous estimation method. A technique for calculating the model residuals is given in Section 10.4.4, so that the statistical properties of the residuals can be tested by specified diagnostic checks. If, for example, the residuals fail to pass the whiteness criterion, another type of model should be chosen in order to satisfy this important modelling assumption. Next, a procedure is presented in Section 10.4.5 for calculating minimum mean square error (MMSE) forecasts for a FGN model. Following this, an exact simulation procedure is given in Section 10.4.6 for simulating FGN. This new simulation method eliminates the need for approximating FGN by other types of stochastic processes.

The FGN model is an example of what is called a long memory process defined in Section 2.5.3. On the other hand, the ARMA models of Chapter 3 possess short memory. For discriminating between the long memory FGN models and the short memory ARMA models, the Akaike information criterion (AIC) defined in Section 6.3.2 can be employed. For the six annual riverflow time series considered in Section 10.4.7, the AIC selects the ARMA model in preference to the FGN model in each case.

To investigate statistical properties of the RAR and the *Hurst coefficient K*, simulation experiments are carried out in Section 10.5. Within Section 10.5.2, simulation studies are executed using white noise while in Section 10.5.3 the simulation studies involve synthetic data generated from both long and short memory models.

A major challenge in stochastic hydrology is to determine time series models that preserve important historical statistics such as the RAR, or equivalently, the Hurst coefficient K. By following the identification, estimation, and diagnostic check stages of model development, ARMA models are determined for 23 geophysical time series in Section 10.6. Simulation studies are then performed to determine the small sample *empirical cumulative distribution function (ECDF)* of the RAR or K for various ARMA models. The ECDF for these statistics is shown to be a function of the time series length N and the parameter values of the specific ARMA model being considered. Furthermore, it is possible to determine as accurately as desired the distribution of the RAR or K. A theorem is given to obtain confidence intervals for the ECDF in order to guarantee a prescribed precision. Then it is shown by utilizing simulation results and a given statistical test that ARMA models do preserve the observed RAR or K of the 23 geophysical time series. Consequently, *ARMA models provide an explanation for the Hurst phenomenon.* Finally, various estimates for the Hurst coefficient are estimated and compared in Section 10.7 for the 23 given time series.

The FGN model defined in this chapter is one example of a long memory model. Another example is the mixed Gamma ARMA(1,1) model proposed by Sim (1987). A flexible class of long memory models based on sound theoretical foundations is the *fractional autoregressive-moving average or FARMA family of models*. In reality, FARMA models constitute direct extensions of ARMA and ARIMA models. Within Chapter 11, FARMA models are defined and model construction techniques are presented.

10.2 DEFINITIONS

The definitions presented in this section reflect long term storage requirements of reservoirs and are needed for explaining the Hurst phenomenon in Section 10.3.1.

Consider a time series $z_1, z_2, \ldots, z_N$. Define the *kth general partial sum* as

$$S'_k = S'_{k-1} + (z_k - \alpha \bar{z}_N) = \sum_{i=1}^{k} z_i - \alpha k \bar{z}_N , \quad k = 1,2,\ldots,N \qquad [10.2.1]$$

where S'_0 equals 0, $\bar{z}_N = \frac{1}{N}\sum_{i=1}^{N} z_i$ is the mean of the first N terms of the series, and α is a constant satisfying $0 \le \alpha \le 1$. The *general (cumulative) range* R'_N is defined as

$$R'_N = M'_N - m'_N \qquad [10.2.2]$$

where $M'_N = \max(0,S'_1,S'_2,\ldots,S'_N)$ is the *general surplus*, and $m'_N = \min(0,S'_1,S'_2,\ldots,S'_N)$ is the *general deficit*. Thus, R'_N is the range of cumulative departures of the random variables $z_1,z_2,\ldots,z_N$, from $\alpha\bar{z}_N$. When random variables such as $z_1,z_2,\ldots,z_N$, are employed in summation operations, they are often referred to as *summands*. The *rescaled general range* R'_N is given as

$$\bar{R}'_N = R'_N / D'_N \qquad [10.2.3]$$

where $D'_N = N^{-1/2}\left[\sum_{i=1}^{N}(z_i - \alpha\bar{z}_N)^2\right]^{1/2}$ is the *general standard deviation*.

The constant α can be thought of as an *adjustment factor*, or in storage theory, it can be interpreted as the degree of development of reservoir design. Two special cases for α are of particular importance in water resources. For $\alpha = 0$ (no adjustment) the kth general partial sum S'_k is replaced by the *crude partial sum* S_k, which is defined by

$$S_k = S_{k-1} + z_k = \sum_{i=1}^{k} z_i , \quad k = 1,2,\ldots,N \qquad [10.2.4]$$

where $S_0 = 0$. The *crude range* R_N is defined analogous to R'_N as

$$R_N = M_N - m_N \qquad [10.2.5]$$

where $M_N = \max(0,S_1,S_2,\ldots,S_N)$ is the *crude surplus*, and $m_N = \min(0,S_1,S_2,\ldots,S_N)$ is the *crude deficit*. Similarly, the *rescaled crude range* is

$$\bar{R}_N = R_N / D_n \qquad [10.2.6]$$

where $D_N = N^{-1/2}\left[\sum_{i=1}^{N} z_i^2\right]^{1/2}$ is the *crude deviation*.

When $\alpha = 1$ (maximum adjustment or development), the *kth adjusted partial sum* S^*_k is given by

$$S^*_k = S^*_{k-1} + (z_k - \bar{z}_N) = \sum_{i=1}^{k} z_i - k\bar{z}_N , \quad k = 1,2,\ldots,N \qquad [10.2.7]$$

where $S^*_0 = 0$ and $S^*_N = 0$. The *adjusted range* R^*_N is defined as

$$R^*_N = M^*_N - m^*_N \qquad [10.2.8]$$

where $M^*_N = \max(0, S^*_1, S^*_2, \ldots, S^*_N)$ is the *adjusted surplus*, and $m^*_N = \min(0, S^*_1, S^*_2, \ldots, S^*_N)$ is the *adjusted deficit*. Finally, the *rescaled adjusted range* is

$$\bar{R}^*_N = R^*_N / D^*_N \qquad [10.2.9]$$

where $D^*_N = N^{-1/2}\left[\sum_{i=1}^{N}(z_i - \bar{z}_N)^2\right]^{1/2}$ is the *sample standard deviation*. Figure 10.2.1 graphically illustrates the concepts of S^*_k, M^*_N, m^*_N, and R^*_N.

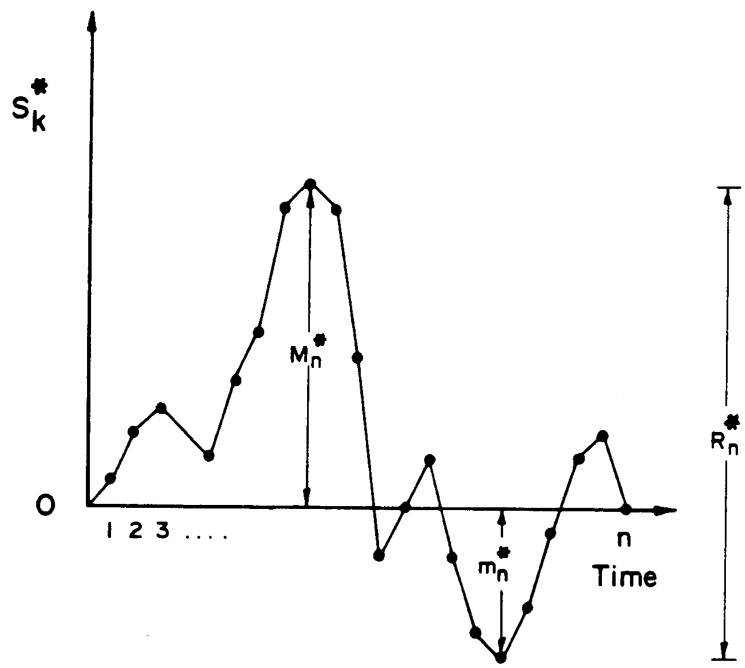

Figure 10.2.1. Adjusted range.

The statistics described in this section are extremely useful in *reservoir design*. If the z_t are average annual volumes of riverflow, then $\sum_{i=1}^{k} z_i$ is the inflow into a reservoir in k years, and $\alpha k \bar{z}_N$ is the outflow at a level of development α. The S'_k in [10.2.1] represents the *storage* after k years. Also, R'_N is the minimum reservoir capacity required to satisfy a constant draft of $\alpha \bar{z}_N$ without experiencing shortages or spills over the period spanned by the inflow sequence $z_1, z_2, \ldots, z_N$. When $\alpha = 1$, the water in the river would be used to its full potential.

The time series $z_1, z_2, \ldots, z_N$ is said to be *covariance stationary* (see Section 2.4.2) if the mean

$$\mu = E[z_t] \qquad [10.2.10]$$

and the theoretical *autocovariance function (ACF)*

$$\gamma_k = E[(z_t - \mu)(z_{t-k} - \mu)] \qquad [10.2.11]$$

both exist and do not depend on t. The statistical properties of any covariance stationary Gaussian time series are completely determined by its mean μ, variance γ_0, and theoretical *autocorrelation function (ACF)*,

$$\rho = \gamma_k / \gamma_0 \qquad [10.2.12]$$

The physical interpretations of the stationary assumptions are discussed in Section 2.4 and also by Klemes (1974).

Often it seems reasonable to assume that recent values of a time series contain more information about the present and future than those in the remote past. Accordingly, it is assumed that the theoretical ACVF is *summable* as defined by (Brillinger, 1975)

$$M = \sum_{k=-\infty}^{\infty} |\gamma_k| < \infty \qquad [10.2.13]$$

As is also mentioned in Section 2.5.3, a covariance stationary time series model is said to have a *short or a long memory* according to whether the theoretical ACVF (or equivalently the theoretical ACF) is summable. Thus, the FGN model has a long memory (for the model parameter H in the range $0.5 < H < 1$), whereas the ARMA models have a short memory. For a specified range of a model parameter d, the FARMA models of Chapter 11 also possess long memory.

10.3 HISTORICAL RESEARCH

10.3.1 The Hurst Phenomenon and Hurst Coefficients

Hurst (1951, 1956) stimulated interest in the RAR statistic by his studies of long-term storage requirements for the Nile River. On the basis of a study of 690 annual time series comprising streamflow, river and lake levels, precipitation, temperature, pressure, tree ring, mud varve, sunspot and wheat price records, Hurst implied that $\bar{R}^*_N$ varies with N as

$$\bar{R}^*_N \propto N^h \qquad [10.3.1]$$

where h is the *generalized Hurst coefficient*. The above equation can be written in the general form

$$\bar{R}^*_N = aN^h \qquad [10.3.2]$$

where a is a coefficient that is not a function of N. It should be noted that Hurst did not explicitly state the generalized Hurst law of [10.3.2] in his research papers. However, by choosing the coefficient a to have a value of $(1/2)^h$, Hurst in effect estimated h by the *Hurst coefficient K* in the empirical equation

$$\bar{R}^*_N = (N/2)^K \qquad [10.3.3]$$

By taking logarithms of [10.3.3], an explicit relationship for K is then

$$K = \frac{\log \bar{R}^*_N}{\log N - \log 2} = \frac{\log R^*_N - \log D^*_N}{\log N - \log 2} \qquad [10.3.4]$$

Employing series that varied in length from 30 to 2000 years, Hurst found K to range from 0.46 to 0.96 with a mean of 0.73 and a standard deviation of 0.09.

Assuming a normally independently distributed (NID) process, Hurst (1951) utilized some coin-tossing experiments to develop the theoretical asymptotic relationship for the expected value of the adjusted range as

$$E[R^*_N] = (\pi N \gamma_0/2)^{1/2}$$

or

$$E[R^*_N]/(\gamma_0)^{1/2} = 1.2533N^{1/2} \qquad [10.3.5]$$

Using the theory of Brownian motion, Feller (1951) rigorously established the above asymptotic formula for any sequence of IID random variables possessing finite variance. It follows from a standard convergence theorem in probability theory (Rao, 1973, p. 122) that for large N,

$$E(\bar{R}^*_N) = 1.2533N^{1/2} \qquad [10.3.6]$$

Even though Hurst studied the RAR for small N and not for the adjusted range, the form of [10.3.5] prompted him to use K in [10.3.4] as an estimate of h and also to assume K to be constant over time. However, for 690 geophysical time series, Hurst found K to have an average of 0.73, while the asymptotic, or limiting, value of K given by [10.3.6] is 0.5. This discrepancy is referred to as the *Hurst phenomenon*. The search for a reasonable explanation of the Hurst phenomenon and the need for methods whereby the statistics related to Hurst's work can be incorporated into mathematical models have intrigued researchers for decades.

In addition to K, other estimates of the generalized Hurst coefficient h in [10.3.2] have been formulated. Based upon the structure of [10.3.6], Gomide (1975, 1978) suggested estimating h by the YH that is given in the following equation:

$$\bar{R}^*_N = 1.2533N^{YH} \qquad [10.3.7]$$

The average value of YH for the 690 series considered by Hurst is 0.57 rather than 0.73.

Siddiqui (1976) proposed a method of evaluating h if the underlying process is assumed to be an ARMA process. The estimate of Siddiqui is the result of a comparison between an asymptotic result for calculating $E(\bar{R}^*_N)$ for ARMA processes and the form of [10.3.2]. Siddiqui's estimate of h and the statistic YH of Gomide (1975, 1978) are calculated in Section 10.7 for the 23 geophysical time series considered in Section 10.6. Appropriate conclusions are drawn regarding the behaviour of these statistics in relationship to K and whether they exhibit the Hurst phenomenon. For the case of a white noise process, Siddiqui's estimate of h is identical with Gomide's statistic YH in [10.3.7].

For NID random variables, Anis and Lloyd (1976) have suggested a specific estimate of h that is a function of the sample size. By taking logarithms of [10.3.2] for the expected value of the RAR, the following equation is obtained.

$$\log E[\bar{R}^*_N] = \log a + h\log N \qquad [10.3.8]$$

Anis and Lloyd (1976) defined the local Hurst exponent $h(N)$ as the derivative

$$h(N) = \partial(\log E[\bar{R}^*_N])/\partial(\log N) \qquad [10.3.9]$$

The exponent $h(N)$ can be tabulated approximately from the equation

$$h(N) = \frac{\log E[\bar{R}^*_{N+1}] - \log E[\bar{R}^*_{N-1}]}{\log(N+1) - \log(N-1)} \qquad [10.3.10]$$

where $E(\bar{R}^*_N)$ is calculated exactly by using the formula of Anis and Lloyd (1976) that is also given in [10.3.15]. It should be noted that previously Salas-La Cruz and Boes (1974) had defined an exponent similar to $h(N)$ for the general range where $0 \leq \alpha \leq 1$.

Because the entries for the expected value of the RAR on the right-hand side of [10.3.10] are calculated directly from a theoretical formula, $h(N)$ is not a function of the data and is, therefore, not a statistic. Nevertheless, it would perhaps be possible to fit some type of stochastic model to a given time series and then to derive the RAR terms in [10.3.10] by using simulation. Most likely, this type of procedure may not be a worthwhile venture, and hence $h(N)$ probably will have limited use in practical hydrological problems.

Anis and Lloyd (1976, p. 115, Table 1) list values of $h(N)$ for N ranging from 5 to 10^6. Although the magnitude of $h(N)$ asymptotically approaches 0.5 for increasing N, at lower values of N, the $h(N)$ is significantly larger than 0.5. For instance, when N possesses values of 5, 40, 100, 200, and 500, then $h(N)$ has magnitudes of 0.6762, 0.5672, 0.5429, 0.5315, and 0.5202, respectively.

In the development of an estimate for the parameter H in FGN models, Mandelbrot and Wallis (1969d) assumed a form of the Hurst law that is identical with [10.3.2]. For a given time series $z_1, z_2, \ldots, z_N$, let $\bar{R}^*_{r'}(t,r)$ denote the RAR of the subseries $z_t, z_{t+1}, \ldots, z_r$, and let $r' = r - t + 1$. When examining scatter plots (or "pox diagrams") of $\log\bar{R}^*_{r'}(t,r)$ versus $\log r'$ for a number of selected values of t and r, Mandelbrot and Wallis (1969d) were using for each subseries a Hurst law given by

$$\bar{R}^*_{r'}(t,r) = a(r')^h \qquad [10.3.11]$$

Wallis and Matalas (1970) have suggested the G Hurst estimator for estimating the parameter H in FGN models and also h in [10.3.11]. This procedure estimates h by calculating the slope of the regression of the averaged values of $\log\bar{R}^*_{r'}(t,r)$ on $\log r'$ for specified values of t and r.

When Hurst originally formulated [10.3.3] there is no doubt that he was attempting to derive an empirical law that would be valid for a wide range of geophysical phenomena. In particular, an equation such as [10.3.3] would be extremely useful for reservoir design if the phenomenon being modelled were average annual riverflows. However, the distribution of K plus the other types of Hurst exponents summarized in this section are a function of the sample size N. For example, the empirical cumulative distribution functions of K for various values of

N for certain types of ARMA processes are given in Section 10.5.3. In addition, as shown in Section 10.6, when K is estimated for 23 given geophysical time series, K seldom has exactly the same value for any given pair of data sets. Because of the aforementioned facts, the empirical law of Hurst in [10.3.3] loses much of its simplicity and also its potential for being a universal law. This inherent lack of universality of Hurst's law may be due to the fact that the general form of [10.3.3] resembles the asymptotic formula given in [10.3.6], whereas in practice it is necessary to deal with small and moderate sample sizes.

Because the RAR possesses many attractive statistical features, Hurst perhaps should have concentrated his efforts on studying the properties of $\bar{R}^*_N$ rather than those of K. The RAR statistic is independent of the magnitude of the mean level and standard deviation of a time series. If the data are modelled by an ARMA process, $\bar{R}^*_N$ is only a function of the sample size N and the autoregressive (AR) and moving average (MA) parameters and is independent of the variance of the innovations (Hipel, 1975, Appendix B). From [10.3.4] it can be seen that K is simply a transformation of $\bar{R}^*_N$ and, therefore, also possesses the aforementioned properties of the RAR. Nevertheless, the formulation of K in [10.3.4] as a function of $\bar{R}^*_N$ only introduces an unnecessary transformation and does not give K any additional advantageous statistical properties that are not already possessed by the RAR. It is therefore recommended that future research should concentrate on the RAR rather than on the various types of Hurst exponents discussed in this section.

Since the concept of the Hurst coefficient is so entrenched in the literature, it is widely quoted in the remainder of this chapter. However, the reader should be aware that the statistic of primary concern is the RAR. Even the use of the G Hurst statistic (Wallis and Matalas, 1970), which was primarily developed as an estimate for the parameter H in FGN models, is questionable. It is demonstrated later in this chapter that a MLE of H is a more efficient procedure to employ.

10.3.2 The Hurst Phenomenon and Independent Summands

Besides the results of Feller (1951), Hurst's work influenced other researchers to develop theoretical derivations for statistics related to the cumulative range. Because of the mathematical complexity in deriving theoretical formulae for the moments of statistics connected with the range, a large portion of the research was devoted to the special case of independent summands. Anis and Lloyd (1953) developed a formula for the expected value of the crude range for standard NID variates. Anis (1955) derived the variance of M_N and subsequently a method for obtaining all the moments of M_N (Anis, 1956).

Solari and Anis (1957) determined the mean and variance of the adjusted range for a finite number of NID summands. Feller (1951) had noted that the sampling properties of the adjusted range were superior to those of the crude range. The results of Solari and Anis (1957) for the variance of M^*_N substantiated the conclusion of Feller when this variance was compared to that of M_N (Anis, 1956).

Moran (1964) initiated a new line of development when he observed that the expected value of cumulative ranges could easily be derived from a combinatorial result known as Spitzer's lemma. He showed that for moderate N, distributions with very large second moments about the mean could cause the $E(M_N)$ to increase more quickly than $N^{1/2}$. This, in turn, implied that the crude range would do likewise.

For independently, stably distributed summands with the characteristic exponent ν, Boes and Salas-La Cruz (1973) showed that asymptotically

$$E(\bar{R}^*_N) \propto N^{1/\nu} \qquad [10.3.12]$$

where $1 < \nu \leq 2$. The general stable distribution with characteristic exponent ν is defined for $1 < \nu \leq 2$ in terms of its characteristic function

$$\omega(u) = E(e^{iuz_t}) \qquad [10.3.13]$$

by

$$\log\omega(u) = i\mu u + \sigma^2 |u|^\nu \left\{ 1 + i\beta(u/|u|)\tan[(\pi/2)\nu] \right\} \qquad [10.3.14]$$

where $i = (-1)^{1/2}$, μ is the location parameter for the random variable z_t; σ is the scale parameter for the random variable z_t; and β is the measure of skewness. For $\beta = 0$ and $\nu = 2$, the normal distribution is obtained. Stable distributions with characteristic exponent $1 < \nu < 2$ generate more extreme observations than the normal distribution. Granger and Orr (1972) have suggested that economic time series are best modelled by a stable distribution with characteristic exponent $1.5 < \nu < 2$. From [10.3.12] it could be suggested that a stable distribution with $\nu = 1.37$ (approximately) for geophysical time series could explain Hurst's findings. However, because for the case of stable distributions with $1 < \nu < 2$ the sample variance is not a consistent estimator of the scale parameter σ, it does not follow that [10.3.12] will hold for the RAR. In fact, simulation experiments reported later in this chapter show that the expected value of the RAR for independently stably distributed summands with characteristic exponent $\nu = 1.3$ very nearly equals the expected value of the RAR for NID summands.

All of the aforesaid research was influenced by the original work of Hurst. However, mathematicians have for a long time been investigating the crude range of independent summands independently of Hurst's empirical research. Anis and Lloyd (1976) give a brief survey of mathematical studies of the crude range. Further references can also be found in a paper by Berman (1964).

Unfortunately, none of the foregoing theoretical investigations discussed in this section have dealt with the RAR. However, for a NID process, Anis and Lloyd (1976) have successfully proved the following exact equation to be the expected value of the RAR:

$$E(\bar{R}^*_N) = \frac{\Gamma[\frac{1}{2}(N-1)]}{(\pi)^{1/2}\Gamma(\frac{1}{2}N)} \sum_{r=1}^{N-1} \frac{N-r}{r} \qquad [10.3.15]$$

10.3.3 The Hurst Phenomenon and Correlated Summands

Introduction

When Hurst (1951) theoretically derived [10.3.5] for the adjusted range, he assumed normality of the process, he developed that equation as an asymptotic relationship relationship, and he assumed independence of the time series. As was pointed out by Wallis and Matalas (1970), these three facts respectively caused the following three possible explanations of the Hurst phenomenon: (1) nonnormality of the probability distribution underlying the time series, (2) transience (i.e., N is not large enough for the Hurst coefficient to attain its limiting value of 0.5), and (3) autocorrelation due to nonindependence.

For independent summands, nonnormality of the underlying process has largely been discounted as a possible explanation of the Hurst phenomenon. If a very large sample is being considered, the asymptotic expression in [10.3.6] has been shown to be valid for IID random variables. For samples of small and moderate lengths, simulation studies later in this chapter (see Table 10.5.1) reveal that the RAR is very nearly independent of the distribution of the random variables. Because the Hurst coefficient K is definitely a function of N for independent summands (see, for example, Table 10.5.2 for the NID case), then transience constitutes a plausible explanation to Hurst's dilemma (also see Salas et al. (1979)).

For the autocorrelated case, Wallis and O'Connell (1973) correctly concluded that transience is obviously connected with the autocorrelation structure of the generating process, and, therefore, these two effects must be considered simultaneously when attempting to account for the Hurst phenomenon. As is illustrated by simulation studies in Sections 10.5 and 10.6 for ARMA models, both transience and autocorrelation form an explanation of the Hurst phenomenon. In this section, the roles of both short memory and long memory processes for explaining and modelling the Hurst phenomenon are examined.

Hurst (1951) actually conjectured that K had a value of 0.73 and not 0.5 because of persistence. This is the tendency for high values to be followed by high values and low values by low values which are referred to by Mandelbrot and Wallis (1968) as the Noah and Joseph effects, respectively. Persistence is caused by the dependence of naturally occurring time series as exhibited in their serial correlation structure. For reservoir design this means that for a given value of N the size of a reservoir that releases the mean flow each year would need to be larger than the capacity corresponding to an uncorrelated series of inflows.

Short Memory Models

Barnard (1956) and Moran (1959) observed that for the standard short memory time series models the following asymptotic formula is valid:

$$E(\bar{R}^*_N) = aN^{1/2} \qquad [10.3.16]$$

where a is a coefficient that does not depend on N. Mandelbrot and Van Ness (1968) proved that for large N, [10.3.16] holds for any short memory time series model. Siddiqui (1976) demonstrated that for any model with a summable theoretical ACVF,

$$a = \left(\frac{\pi}{2\gamma_0} \sum_{i=-\infty}^{\infty} \gamma_i \right)^{1/2} \qquad [10.3.17]$$

It has been argued by some authors that because short memory models, such as the ARMA processes, imply a limiting value of K equal to 0.5 and since the observed K in annual geophysical time series is about 0.7, short memory models are not appropriate models for synthetic streamflow generation. It should therefore be emphasized that asymptotic results are only relevant in that they provide an approximation to the exact results for the true (finite) series length.

Anis and Lloyd (1976) showed that [10.3.15] also holds exactly for symmetrically correlated normal summands. But such a time series has a long memory, since its theoretical ACVF is not summable. Because [10.3.15] is also valid for short memory NID random variables, this fact provides a counterexample to the claim of some researchers that long memory models are a necessary explanation of the Hurst phenomenon. Conversely, Klemes (1974) has shown that zero memory nonstationary models could produce the Hurst phenomenon. By simulation experiments with white noise, he varied the mean level in different manners and showed how K increased in value due to this type of nonstationarity. Klemes also demonstrated by simulation that random walks with one absorbing barrier, which often arise in natural storage systems, could cause the RAR to have certain properties related to the Hurst phenomenon.

Hurst (1957) was the first scientist to suggest that a nonstationary model in which the mean of the series was subject to random changes could account for higher values of the Hurst coefficient K and hence the Hurst phenomenon. Similar models have been studied by Klemes (1974) and Potter (1976). As generalizations of the models of Hurst (1957), Klemes (1974) and Potter (1976), the *shifting level processes* were developed by Boes and Salas (1978). Further research in shifting level processes is provided by Salas and Boes (1980), Ballerini and Boe (1985), and Smith (1988). The basic idea underlying a shifting level process is that the level of the process randomly shifts to different levels which last for random time periods as the process evolves over time.

In a four page commentary, D'Astous et al. (1979) demonstrated that the annual precipitation data employed by Potter (1976) may not justify the concept of a shifting level time series. Using simulation and the segmentation scheme suggested by Potter (1976) for isolating shifting levels, they showed that an ARMA(1,1) process can mimic this type of changing level. If the mean level of a time series changes due to known natural or human intervention, then the intervention model of Chapter 19 can be used to model the data.

Matalas and Huzzen (1967) performed statistical experiments to determine whether K is preserved by Markov models. For values of the lag 1 autocorrelation coefficient ρ_1 ranging from 0 to 0.9, they calculated the $E(K)$ based upon 10^4 simulations for particular values of N and ρ_1. For values of N and ρ_1, compatible with what occurs in annual riverflows if those flows are assumed Markov, they found K to have an average of about 0.7. Because a mean of approximately 0.7 for K occurs in natural time series, they implied that perhaps the small sample properties of K are preserved by a Markov model. Nevertheless, a later simulation study of Wallis and Matalas (1970) suggested that the observed sample lag 1 autocorrelations for flows in the Potomac River basin were too low for a first order AR process adequately to preserve the Hurst K. However, a Markov model may not necessarily be the best short memory model to fit to a given time series. Rather, it is recommended to select the proper ARMA model by adhering to the identification, estimation, and diagnostic check stages of model construction, as explained in Part III of this book. In some cases, the appropriate model may indeed be a Markov model. In Section 10.6, it is demonstrated that, for 23 geophysical time series ranging in length from $N = 96$ to $N = 1164$, properly fitted ARMA models do adequately preserve K.

Several other authors have also suggested that short memory models may preserve K. Gomide (1975) has completed further simulation studies of the RAR for Markov models. O'Connell (1974a,b) advocated employing an ARMA(1,1) model to approximate the long memory FGN model and thereby perhaps to preserve K. To accomplish this, the AR parameter must have a value close to unity, so that the ACF of the process will attenuate slowly and hence approximate the theoretical ACF of the FGN process. In practice, this approach may not be viable. The proper ARMA model that is fit to the data may not be ARMA(1,1), and even if it is ARMA(1,1), an efficient MLE of the parameters may not produce an estimate of the AR parameter that is close to 1. This parameter estimation problem is acknowledged by O'Connell (1976). In addition, it is no longer necessary to approximate FGN by a short memory model such as an ARMA(1,1) model because as is shown in Section 10.4.6 it is now possible to simulate FGN exactly.

Long Memory Models

A long memory model known as FGN was introduced into the hydrological literature by Mandelbrot and Wallis (1968, 1969a to e) to explain the Hurst phenomenon. In Section 10.4, the FGN model is defined and new developments in FGN modelling are presented. Other research on stochastic processes related to FGN is given by authors such as Taqqu (1979) and Cox (1984). In Chapter 11, the theory and practice of the long memory FARMA class of models is presented.

10.4 FRACTIONAL GAUSSIAN NOISE

10.4.1 Introduction

The connection between FGN and Hurst's law is the parameter H in FGN that is often estimated using the Hurst coefficient K in [10.3.4]. The FGN model was first proposed by Mandelbrot (1965), and a mathematical derivation was given by Mandelbrot and Van Ness (1968) and Mandelbrot and Wallis (1969c). The literature concerning the FGN model has been summarized by authors such as Wallis and O'Connell (1973), O'Connell (1974b, Ch. 2), Hipel (1975), Lawrance and Kottegoda (1977) and McLeod and Hipel (1978a). Consequently, only the main historical points of practical interest are discussed in Section 10.4.2. Following a brief historical description and definition of FGN in the next section, new advancements are presented. These include efficient parameter estimation, model diagnostic checking, forecasting, and exact simulation. In an application section, FGN models are compared to ARMA models when both types of models are fitted to six average annual riverflow series.

10.4.2 Definition of FGN

In the development of FGN processes, Mandelbrot (1965) considered a continuous time process $B_H(t)$ that satisfies the self-similarity property such that for all τ and $\varepsilon > 0$, $B_H(t+\tau) - B_H(t)$ has exactly the same distribution as $[B_H(t+\tau\varepsilon) - B_H(t)]/\varepsilon^H$. It can be shown that the sequential range of $B_H(t)$ will increase proportionally to N^H, where the sequential range is defined by

$$\text{sequential range} = \max_{t<r<t+N} B_H(r) - \min_{t<r<t+N} B_H(r) \qquad [10.4.1]$$

where t is continuous time and H is the model parameter. When the process $B_H(t)$ is Gaussian, it is called fractional Brownian motion. Discrete time *fractional Gaussian noise (FGN)* is defined for discrete time t by the increments

$$z_t = B_H(t+1) - B_H(t) \qquad [10.4.2]$$

FGN is what Mandelbrot and Wallis (1969c) consider to be a model of Hurst's geophysical time series.

Mandelbrot and Van Ness (1968) and Mandelbrot and Wallis (1969a,b,c) have derived a number of properties of FGN. First, the parameter H must satisfy $0 < H < 1$. The sample mean and variance of FGN are consistent estimators of the true mean and variance, and FGN is covariance stationary. The expected values of the crude and adjusted ranges for FGN are the asymptotic relationships

$$E(R_N) = a_H N^H, \quad 0 < H < 1 \qquad [10.4.3]$$

and

$$E(R^*_N) = b_H N^H, \quad 0 < H < 1 \qquad [10.4.4]$$

where a_H and b_H are coefficients that do not depend on N. It can also be shown that for large N (Rao, 1973, p. 122),

$$E(\bar{R}^*_N) = aN^H \qquad [10.4.5]$$

Although the above asymptotic formulae are correct mathematically, they may possess limitations with respect to modelling Hurst's findings. Of foremost importance is the fact that Hurst examined $\bar{R}^*_N$ for small N and not the asymptotic expected values of R_N, R^*_N, and $\bar{R}^*_N$. Behaviour of any of the range statistics for large N does not necessarily infer the structure of $\bar{R}^*_N$ for small and moderate N. Even though [10.4.3] to [10.4.5] are asymptotically valid, in reality the Hurst coefficient is a function of N and is not a constant as is the parameter H in FGN. For example, as is shown by simulation experiments for NID random variables in Table 10.5.2, the expected value of the Hurst coefficient K is significantly larger than 0.5 for small N. A sequence of NID random variables is equivalent to a FGN process with $H = \frac{1}{2}$.

The theoretical ACF at lag k of FGN is given by

$$\rho_k = \frac{1}{2}[(k+1)^{2H} - 2k^{2H} + (k-1)^{2H}], \quad 0 < H < 1 \text{ and } k \geq 1 \tag{10.4.6}$$

For large lags, [10.4.6] may be approximated by

$$\rho_k = H(2H-1)k^{2H-2} \tag{10.4.7}$$

The $N \times N$ correlation matrix for FGN is given as

$$\mathbf{C}_N(H) = [\rho_{|i-j|}] \tag{10.4.8}$$

where $\rho_0 = 1$ and ρ_k is calculated from [10.4.6] for $k \geq 1$. The Cholesky decomposition (Healy, 1968) of $\mathbf{C}_N(H)$ is determined such that

$$\mathbf{C}_N(H) = \mathbf{M}\mathbf{M}^T \tag{10.4.9}$$

where $\mathbf{M}$ is the $N \times N$ lower triangular matrix having m_{ij} as a typical element. The matrix $\mathbf{M}$ is used for carrying out diagnostic checks, and simulating with FGN in Sections 10.4.4 and 10.4.6, respectively.

An examination of [10.4.6] and [10.4.7] reveals that $\rho_k \to 0$ as $k \to \infty$, but ρ_k is not summable if $\frac{1}{2} < H < 1$. Therefore, for $\frac{1}{2} < H < 1$, the FGN process has long memory. When $0 < H \leq \frac{1}{2}$, FGN constitutes a short memory process.

For many geophysical phenomena, the estimates for H are greater than $\frac{1}{2}$ but less than 1. Because the FGN model is not summable for H in this range, the statistical effect of past events on present behaviour attenuates very slowly. Therefore, long term persistence, as described by the theoretical ACF, is synonymous with $\frac{1}{2} < H < 1$. Some hydrologists claim that the form of the theoretical ACF for $\frac{1}{2} < H < 1$ is explained by the physical existence of an extremely long memory in hydrologic and other processes. But, as was pointed out by Klemes (1974), making inferences about physical features of a process based on operational models can be not only inaccurate but also misleading. Klemes correctly states that "... it must be remembered that the mathematical definition of FGN did not arise as a result of the physical or dynamic properties of geophysical and other processes but from a desire to describe an observed geometric pattern of historic time series mathematically... Thus FGN is an operational, not a physically founded model." Klemes demonstrates that the Hurst phenomenon could be due to zero memory nonstationary models and also specific types of storage systems. However, although physical interpretations that use operational models should be formulated and interpreted with caution, one criterion that is essential is that the statistical properties of any historical time series be incorporated properly into the stochastic model.

The appropriateness of long memory processes for modelling annual riverflow and other types of natural time series has been questioned previously by various hydrologists (Scheidegger, 1970; Klemes, 1974). Moreover, later in Section 10.4.7, it is shown that the FGN model can fail to provide an adequate statistical fit to historical annual riverflows.

The FGN model for a time series z_t, $t = 1,2,...,N$, can be specified in terms of the three parameters μ, γ_0, and H, where $E[z_t] = \mu$, $Var[z_t] = \gamma_0$, and the theoretical ACF of z_t is given by [10.4.6]. From these specifications, improved estimation and simulation procedures can be developed. Complete Fortran computer algorithms for these methods are given by Hipel and McLeod (1978b).

As explained in Part III in the book, when determining a long memory or a short memory model or in general any type of stochastic process for modelling a given data set, it is recommended to adhere to the three stages of model development. The first step consists of identifying, or choosing, the type of model to fit to the time series. If circumstances warrant the employment of a FGN process, then at the estimation stage, efficient MLE's of the model parameters can be procured by using the technique developed in Section 10.4.3. It is also shown in Section 10.4.4 how the model residuals of FGN can be calculated after the model parameters have been estimated. If diagnostic checks of the residuals reveal that modelling assumptions such as residual whiteness, normality, and homoscedasticity (i.e., constant variance) are not satisfied, then appropriate action can be taken. For example, a Box-Cox transformation (see Section 3.4.5) of the data prior to fitting a FGN process may rectify certain anomalies in the residuals. In some cases, a short memory model such as an ARMA process may provide a better statistical fit while at the same time preserve important historical statistics such as the RAR. The AIC (see Section 6.3) is recommended as a means of selecting the best model from a set of tentative models that may consist of both short memory and long memory processes.

10.4.3 Maximum Likelihood Estimation

In addition to the mean and variance, an estimate of the parameter H forms the only link that a FGN model has with the real world as represented by the historical data. Previously, various estimates for H have been formulated. Some researchers employ K in [10.3.4] as an estimate of H. Wallis and Matalas (1970) recommend the G Hurst statistics as an estimate of H. Unfortunately, little is known about the theoretical distribution of this estimate, and the G Hurst statistic in effect constitutes only an ad hoc method of calculating H. Young and Jettmar (1976, p. 830, equation 4) suggest a moment estimate for H based on an estimate of the historical ACF at lag 1 and [10.4.6]. They also develop a least squares estimate for H that is formulated by using the sample ACF and [10.4.6] (Young and Jettmar, 1976, p. 831, equation 6). However, McLeod and Hipel (1978b) question the theoretical basis and efficiency of Young and Jettmar's least squares estimate for H.

An alternative approach to estimating the parameters of a FGN model is to employ maximum likelihood estimation. The method of maximum likelihood procedure is widely used for the estimation of parametric models, since it often yields the most efficient estimates (see Section 6.2). Dunsmuir and Hannan (1976) show that the MLE's of the parameters of time series models often yield optimal estimates under very general conditions, which include the FGN model as a special case.

Given a historical time series $z_1, z_2, \ldots, z_N$, the log likelihood of μ, γ_0, and H in the FGN model is

$$\log L^*(\mu,\gamma_0,H) = -\frac{1}{2}\log|C_N(H)| - (2\gamma_0)^{-1}S(\mu,H) - (N/2)\log\gamma_0 \quad [10.4.10]$$

where $C_N(H)$ is the correlation matrix given by [10.4.8]. The function $S(\mu,H)$ in [10.4.10] is determined by

$$S(\mu,H) = (\mathbf{z} - \mu\mathbf{1})^T[C_N(H)]^{-1}(\mathbf{z} - \mu\mathbf{1}) \quad [10.4.11]$$

where $\mathbf{z}^T = (z_1, z_2, \ldots, z_N)$ is a $1\times N$ vector and $\mathbf{1}^T = (1,1,\ldots,1)$ is a $1\times N$ vector. For fixed H, the MLE of μ and γ_0 are

$$\hat{\mu} = \{\mathbf{z}^T[C_N(H)]^{-1}\mathbf{1}\}/\{\mathbf{1}^T[C_N(H)]^{-1}\} \quad [10.4.12]$$

and

$$\hat{\gamma}_0 = N^{-1}S(\hat{\mu},H) \quad [10.4.13]$$

Thus, the maximized log likelihood function of H is

$$\log L_{max}(H) = -\frac{1}{2}\log|C_N(H)| - N/2\log[S(\hat{\mu},H)]/N \quad [10.4.14]$$

The inverse quadratic interpolation search method can then be used to maximize $\log L_{max}(H)$ to determine $\hat{H}$, the MLE of H. The variance of $\hat{H}$, given by $Var(\hat{H})$, is approximately

$$Var(\hat{H}) = -1\left|\frac{\partial^2 \log L_{max}(H)}{\partial H^2}\right|_{H=\hat{H}} \quad [10.4.15]$$

The variance in [10.4.14] can be evaluated by numerical differentiation. If the computer algorithms given by Hipel and McLeod (1978b) are utilized, the computer time required for these calculations is not excessive provided that N is not too large (not larger than about 200). The standard error (SE) of the MLE of H is simply the square root of $Var(\hat{H})$ in [10.4.15].

In order to compare the statistical efficiency of the maximum likelihood and G Hurst estimation procedures, a simulation study is performed. For $H = 0.5$, 0.6, 0.7, 0.8, and 0.9 and for $N = 50$ and 100, 500 simulated series for each FGN model are generated by using the exact simulation technique given in Section 10.4.6. For each synthetic trace, the MLE $\hat{H}$ and the G Hurst estimate obtained by using $GH(10)$ as defined by Wallis and Matalas (1970) are determined. Because $GH(10)$ and $\hat{H}$ are functionally independent of the mean and variance, it is simplest to set the mean equal to zero and to assign the variance a value of unity when generating the synthetic data by using the method of Section 10.4.6. The *mean square errors (MSE)* of the maximum likelihood and $GH(10)$ estimators for a particular value of N are

$$MSE_{MLE}(H,N) = \frac{1}{500}\sum_{i=1}^{500}(\hat{H}_i - H)^2 \quad [10.4.16]$$

and

$$MSE_{GH}(H,N) = \frac{1}{500}\sum_{i=1}^{500}(GH_i - H)^2 \qquad [10.4.17]$$

where $\hat{H}_i$ is the MLE of H for the ith simulated series of length N having a particular true value of H and GH_i is the magnitude of $GH(10)$ for the ith simulated series of length N with a specified true value of H.

The MSE criterion constitutes a practical overall measure for assessing the accuracy of an estimate. The MSE is equal to the square of the *bias* of the estimate plus the variance of the estimate. Because a biased estimator may in certain cases have smaller overall MSE, the "unbiasedness" of an estimate alone is not necessarily the most important requirement of an estimate. The relative efficiency (RE) of the $GH(10)$ estimate in comparison with the MLE $\hat{H}$ is

$$RE(H,N) = [MSE_{MLE}(H,N)]/[MSE_{GH}(H,N)] \qquad [10.4.18]$$

The entries in Table 10.4.1 confirm that the MLE procedure is significantly more accurate than the G Hurst method.

Table 10.4.1. Percentage relative efficiency of $GH(10)$ versus $\hat{H}$.

	N	
H	50	100
0.5	48	38
0.6	55	44
0.7	59	47
0.8	57	43
0.9	50	34

As explained in Section 6.3, the AIC is useful for discriminating among competing parametric models (Akaike, 1974). For the FGN model, the AIC is given by

$$AIC = -2\log L_{max}(H) + 4 \qquad [10.4.19]$$

When comparing models, the one with the smallest AIC provides the best statistical fit with the minimum number of model parameters.

10.4.4 Testing Model Adequacy

After fitting a statistical model to data, it is advisable to examine the chosen model for possible inadequacies which could seriously invalidate the model. The residuals of the FGN model with parameters μ, γ_0, and H can be defined by

$$\mathbf{e} = \mathbf{M}^{-1}(\mathbf{z} - \mu\mathbf{1}) \qquad [10.4.20]$$

where $\mathbf{e}^T = (e_1, e_2, \ldots, e_N)$ is the vector of model residuals. If the chosen model provides an adequate fit, the elements of $\mathbf{e}$ should be white noise that is NID(0,1). Accordingly, for any proposed FGN a suggested diagnostic check is to test the residuals for whiteness by employing suitable tests for whiteness (see Sections 7.3 and 2.6). For instance, the cumulative periodogram test of Section 2.6 could be utilized to check for residual whiteness. Other appropriate tests could be invoked to test whether the less important assumptions of normality (see Section 7.4) and homoscedasticity (see Section 7.5) of the residuals are also satisfied.

10.4.5 Forecasting with FGN

In Section 8.2, minimum mean square error (MMSE) forecasts are defined for use with ARMA models. One can, of course, also determine MMSE forecasts for FGN models. More specifically, Noakes et al. (1988) develop a formula for calculating one step ahead forecasts for a FGN model by employing the standard regression function (Anderson, 1958). First, to obtain the covariance matrix, Γ_N, one can substitute the MLE for H from Section 10.4.3 into [10.4.6] and then divide [10.4.8] by the estimated variance from [10.4.13]. The one step ahead forecast is then given by

$$E\{Z_{N+1}|\mathbf{Z}_N\} = \mu + \gamma_N^T \Gamma_N^{-1}(\mathbf{Z}_N - \mu\mathbf{1}) \qquad [10.4.21]$$

where $\mathbf{Z}'_N = (z_1, z_2, \ldots, z_N)$, and $\gamma_N = (\gamma_N, \gamma_{N-1}, \ldots, \gamma_1)$. Rather than inverting Γ_N, let

$$\Gamma_N \mathbf{X}_N = (\mathbf{Z}_N - \mu\mathbf{1}) \qquad [10.4.22]$$

and solve for X_N. The solution of this system of equations is obtained using a Cholesky decomposition (Healy, 1968) of Γ_N such that

$$\mathbf{M}\mathbf{M}'\mathbf{X}_N = (\mathbf{Z}_N - \mu\mathbf{1}) \qquad [10.4.23]$$

where $\mathbf{M}$ is a $N \times N$ lower triangular matrix. The one step ahead forecast of Z_{N+1} is thus

$$E\{Z_{N+1}|\mathbf{Z}_N\} = \mu + \gamma_N \mathbf{X}_N \qquad [10.4.24]$$

Successive one step ahead forecasts can be obtained using the following procedure. Given $\mathbf{M}$, the covariance matrix for $\mathbf{Z}_{N+1}$ may be written as

$$\Gamma_{N+1} = \begin{bmatrix} \Gamma_N & \gamma_N \\ \gamma_N^T & \gamma_0 \end{bmatrix} = \begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{a}_T & \alpha \end{bmatrix} \begin{bmatrix} \mathbf{M}' & \mathbf{a}' \\ \mathbf{0} & \alpha \end{bmatrix} = M^*M^{*\prime} \qquad [10.4.25]$$

where $\mathbf{M}^*$ is a $(N+1)\times(N+1)$ lower triangular matrix. The Cholesky decomposition of Γ_{N+1} is calculated by noting that

$$\mathbf{M}\mathbf{a} = \gamma_N \qquad [10.4.26]$$

and

$$\alpha = \sqrt{\gamma_0 - \mathbf{a}^T\mathbf{a}}. \qquad [10.4.27]$$

Thus, the forecast of Z_{N+2} is given by

$$E\{Z_{N+2}|\mathbf{Z}_{N+1}\} = \mu + \gamma_{N+1}\mathbf{X}_{N+1} \quad [10.4.28]$$

where $\mathbf{X}_{N+1}$ is obtained from

$$\mathbf{M}^*\mathbf{M}^{*\prime}\mathbf{X}_{N+1} = (\mathbf{Z}_{N+1} - \mu\mathbf{1}) \quad [10.4.29]$$

10.4.6 Simulation of FGN

Historically, researchers have not developed an exact technique for simulating FGN. Instead, short memory approximations of FGN models have been utilized to generate synthetic traces. The methods used for obtaining approximate realizations of FGN include (1) type 1 (Mandelbrot and Wallis, 1969c), (2) type 2 (Mandelbrot and Wallis, 1969c), (3) fast FGN (Mandelbrot, 1971), (4) filtered FGN (Matalas and Wallis, 1971), (5) ARMA(1,1) (O'Connell, 1974a,b), (6) broken line (Rodriguez-Iturbe et al., 1972; Mejia et al., 1972; Garcia et al., 1972; Mandelbrot, 1972), and (7) ARMA-Markov (Lettenmaier and Burges, 1977) models.

Various papers have been written that include surveys and appraisals of one or more of the short memory approximations to FGN (see Lawrance and Kottegoda, 1977; Lettenmaier and Burges, 1977; O'Connell, 1974b; and Wallis and O'Connell, 1973). Although the underlying drawback of all these approximate processes is that the simulated data does not lie outside the Brownian domain (see Mandelbrot and Wallis (1968) for a definition of Brownian domain), additional handicaps of some of the models have also been cited in the literature. For instance, Lawrance and Kottegoda (1977) mention that the lack of a suitable estimation procedure for the parameters of a broken line process is the greatest deterrent to the utilization of that model by hydrologists.

When generating synthetic traces from a short memory approximation to FGN or any other type of stochastic model, proper simulation procedures should be adhered to. If more than one simulated time series from a certain model is needed, then it would be improper to first simulate one long synthetic time series and then to subdivide this longer trace into the required number of shorter time series. Rather, it would be more efficient to generate the shorter series independently so that the resulting estimates from each of the shorter series would be statistically independent. Furthermore, the standard errors of the particular parameters being estimated by the simulation study can be calculated if the estimates are statistically independent, but if they are correlated, the standard errors are not easily estimated. These and other guidelines for use in simulation are discussed in detail in Chapter 9.

Instead of the employment of short memory approximations for simulating FGN, it is possible to generate exact realizations of FGN. This procedure is analogous to the WASIM2 approach for simulating using ARMA models given in Section 9.4 and is based upon a knowledge of the theoretical ACF. Suppose that a FGN series $z_1, z_2, \ldots, z_N$, with parameters μ, γ_0, and H is to be simulated. Firstly, by utilizing an appropriate standard method, generate a Gaussian white noise sequence $e_1, e_2, \ldots, e_N$, that is NID(0,1) [see Section 9.2.3]. Next, calculate the $N \times N$ correlation matrix $\mathbf{C}_N(H)$, using [10.4.8]. Then, the Cholesky decomposition of $\mathbf{C}_N(H)$ is carried out to obtain the lower triangular matrix $\mathbf{M}$ in [10.4.9]. Exact realizations of FGN are calculated from

$$z_t = \mu + \left(\sum_{i=1}^{t} m_{ti} e_i \right) (\gamma_0)^{1/2} \qquad [10.4.30]$$

for $t = 1,2, \ldots, N$, and for $0 < H < 1$, where z_t is the FGN time series value that is $N(\mu,\gamma_0)$, and m_{ti} is from the matrix **M** in [10.4.9]. If the model parameter H is in the range $0.5 < H < 1$, then the synthesized data will lie outside the Brownian domain.

The computer algorithm for exactly simulating FGN is listed in standard Fortran by Hipel and McLeod (1978b). This algorithm requires only about $\frac{1}{2}N(N+2)$ storage locations to simulate a FGN series of length N. Thus, a modest requirement of about 5000 words is required to handle a series of length 100.

10.4.7 Applications to Annual Riverflows

Information concerning six of the longest annual riverflow time series given by Yevjevich (1963) is listed in Table 10.4.2. For each of these time series, the MLE of H in the FGN model and its SE are calculated. Table 10.4.3 lists the MLE and SE's (in parentheses) of H (see Section 10.4.3) and also the Hurst K (see [10.3.4]) and $GH(10)$ (Wallis and Matalas, 1970) estimates for each of the time series.

In Table 10.4.3, notice the difference between the three estimates of the FGN parameter H for each of the data sets. For instance, $\hat{H}$ for the Gota River has a magnitude of 0.839 with a corresponding SE of 0.073. Both the $GH(10)$ and K estimates for the Gota River are more than two times the SE less than the MLE of H.

The parameter estimates for the proper ARMA models that are fitted to the time series in Table 10.4.2 are given later in Table 10.6.3. Both the Danube River and the Rhine River time series are simply white noise. If a time series is NID, the theoretical value of H for a FGN model is 0.5. For both the Danube River and the Rhine River, Table 10.4.3 reveals that the MLE of H is closer to 0.5 than either the $GH(10)$ or the K estimate. In addition, for each of the two data sets, $\hat{H}$ is easily within one SE of 0.5.

In order to determine whether a short memory or a long memory model should be selected for each of the six time series, the AIC can be utilized (see Section 6.3). Table 10.4.4 lists the values of the AIC for the FGN models by using $\hat{H}$ and the best fitting ARMA model. For each of the six cases, the AIC for the ARMA model has a magnitude less than that for the FGN model. Therefore, on the basis of a combination of best statistical fit and model parsimony, the ARMA model should be chosen in preference to the FGN process for the time series considered.

The Gota River is instructive for portraying possible problems that may arise when using FGN models in practice, since it appears that no FGN model can give an adequate fit to this time series. After a FGN model has been fit to a given data set, it is recommended to implement appropriate diagnostic checks for testing model adequacy. It is of utmost importance that the residuals of FGN given by [10.4.20] be white noise. Accordingly, plots of the cumulative periodogram from Section 2.6 for the residuals of the FGN models for the Gota River obtained by using $\hat{H}$, $GH(10)$, and K are displayed in Figures 10.4.1 to 10.4.3, respectively. The 1%, 5%, 10% and 25% significance levels are indicated on the plots. As is shown in the figures, the cumulative periodogram test is significant in all three cases at the 1% level, although the departure from whiteness is not as great for the FGN model when using $\hat{H}$ as it is for the other two

Table 10.4.2. Average annual riverflow time series.

Code Name	River	Location	Period	N
Mstouis	Mississippi	St. Louis, Missouri	1861-1957	96
Neumunas	Neumunas	Smalinikai, U.S.S.R.	1811-1943	132
Danube	Danube	Orshava, Romania	1837-1957	120
Rhine	Rhine	Basle, Switzerland	1807-1957	150
Ogden	St. Lawrence	Ogdensburg, New York	1860-1957	97
Gota	Gota	Sjotorp-Vanersburg, Sweden	1807-1957	150

Table 10.4.3. Estimated statistics for the annual riverflows.

Data Set	$\hat{H}$	$GH(10)$	K
Mstouis	0.674 (0.082)	0.580	0.648
Neumunas	0.591 (0.067)	0.520	0.660
Danube	0.548 (0.063)	0.560	0.633
Rhine	0.510 (0.058)	0.592	0.614
Ogden	0.949 (0.047)	0.868	0.894
Gota	0.839 (0.073)	0.523	0.689

*The parenthetical values are SE's.

Table 10.4.4. AIC values for the fitted FGN and ARMA models.

Data Set	FGN Models	ARMA Models
Mstouis	1400.0	1395.8
Neumunas	1207.5	1198.2
Danube	1666.7	1389.0
Rhine	1531.8	1529.8
Ogden	1176.9	1172.1
Gota	1350.6	1331.0

cases. Therefore, the whiteness diagnostic checks indicate that because of the dependence of the model residuals the FGN processes provide a poor statistical fit to the given data. Hence, it would be advisable to consider another type of process to model the annual riverflows of the Gota River.

When selecting a process to describe a given time series, it is highly desirable that important historical statistics such as the ACF at various lags (especially at low lags for nonseasonal models) be preserved. The inability of the three FGN models for the Gota River to pass the diagnostic check for residual whiteness precludes the preservation of historical statistics by these models. The sample ACF of the Gota River is shown in Figure 10.4.4, while the theoretical ACF of FGN, obtained by using $\hat{H}$ and K, are displayed in Figures 10.4.5 and 10.4.6, respectively. To calculate the theoretical ACF for FGN in Figures 10.4.5 and 10.4.6, the values of $\hat{H}$ and K for the Gota River in Table 10.4.3 are substituted into [10.4.6]. Because the theoretical ACF of FGN obtained by using the $GH(10)$ estimate is not significantly different from 0.5, the plot of this theoretical ACF would be very close to white noise and is, therefore, not given. Nevertheless, comparisons of Figure 10.4.4 with Figures 10.4.5 and 10.4.6 reveal visually that the historical sample ACF is not preserved by the FGN models.

In contrast to the inability of a FGN process to model the Gota riverflows, an ARMA model does provide an adequate fit to the data. By following the identification, estimation, and diagnostic check stages of model construction presented in Part III, the best type of ARMA model to describe the Gota riverflows is an ARMA process with two AR parameters (denoted by ARMA(2,0)). The ARMA(2,0) process provides a slightly better fit than an ARMA model with one MA parameter (denoted as ARMA(0,1)). The AIC also selects the ARMA(2,0) model in preference to the ARMA(0,1) model. In addition, the ARMA(2,0) model passes rigorous diagnostic checks for whiteness, homoscedasticity, and normality of the model residuals.

By knowing the parameter estimates of an ARMA model, it is possible to calculate the theoretical ACF by employing a technique described in Appendix A3.2. Figure 10.4.7 is a plot of the theoretical ACF for the ARMA(2,0) model for the Gota River data. A comparison of Figures 10.4.7 and 10.4.4 demonstrates that the ARMA model preserves the historical ACF especially at the important lower lags. Notice that the value of the ACF for lags 1 to 4 are almost identical for these two plots.

In addition to the use of graphical aids to determine whether historical statistics are preserved, a more rigorous procedure can be followed. In Section 10.6 a statistical test is used in conjunction with Monte Carlo techniques in order to determine the ability of a class of models to preserve specified historical statistics. It is demonstrated that ARMA processes preserve the RAR or equivalently K. This procedure could also be adopted for statistics such as various lags of the ACF to show quantitatively whether or not these statistics are preserved by the calibrated models.

The inability of a FGN process to preserve the ACF and perhaps other historical statistics in some practical applications could be due to the inherent mathematical structure and underlying properties that were discussed previously. Another obvious drawback of FGN is the dependence of the model on only a few parameters. In addition to the mean and variance, an estimate of the parameter H forms the only actual link between the theoretical model and the real world as presented by the data. This renders FGN processes highly inflexible. On the other hand, in ARMA modelling the form of the model is tailored specifically to fit a given set of data. At the

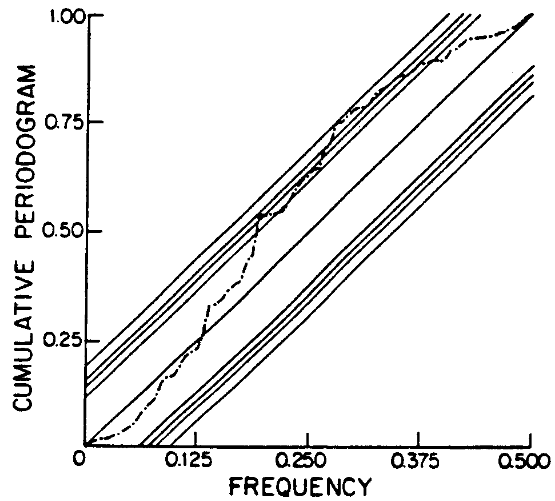

Figure 10.4.1. Gota River residual cumulative periodogram for the FGN model using $\hat{H}$.

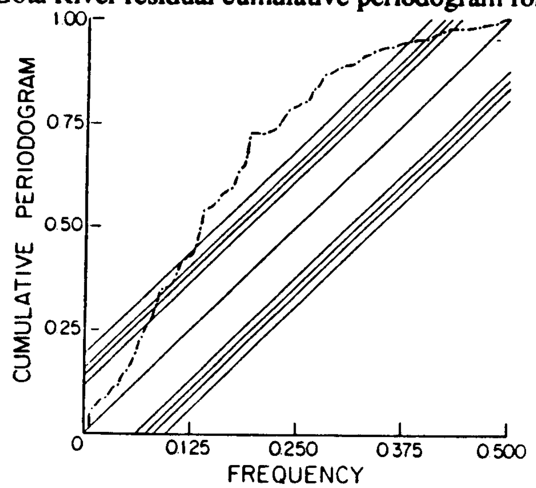

Figure 10.4.2. Gota River residual cumulative periodogram for the FGN model using $GH(10)$.

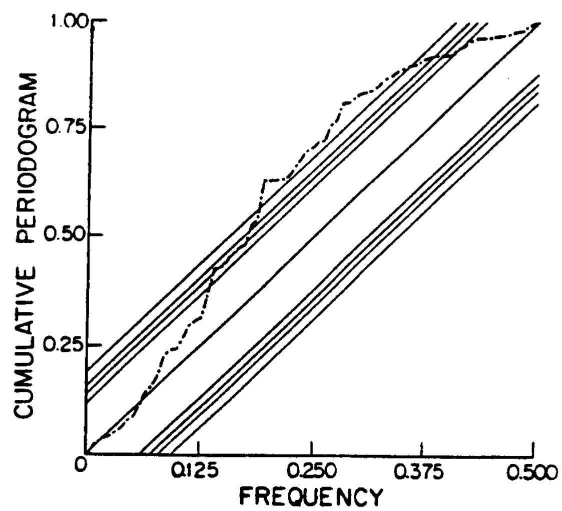

Figure 10.4.3. Gota River residual cumulative periodogram for the FGN model using K.

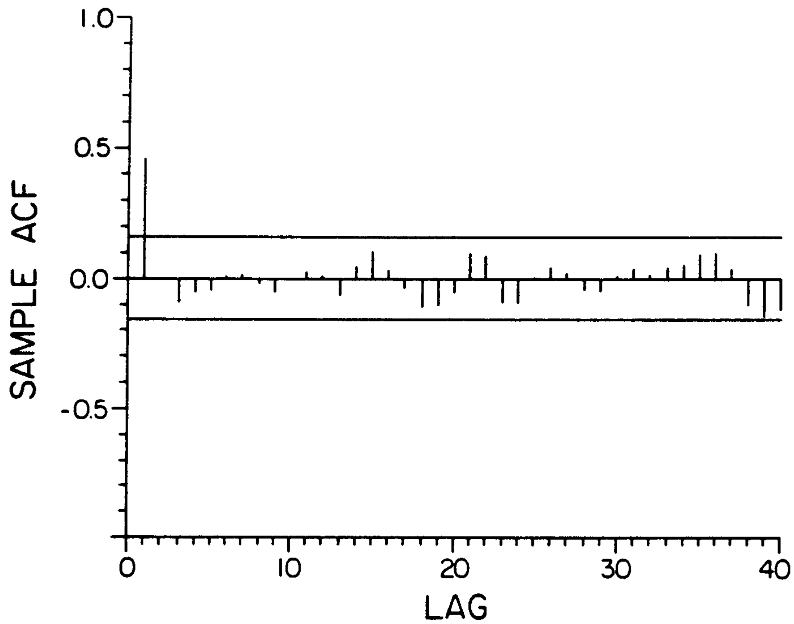

Figure 10.4.4. Sample ACF of the Gota River.

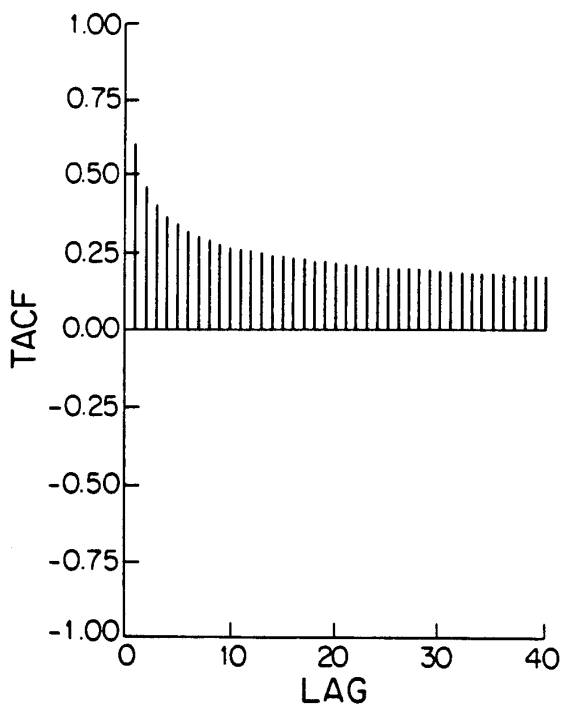

Figure 10.4.5. Theoretical ACF of the Gota River for the FGN model using $\hat{H}$.

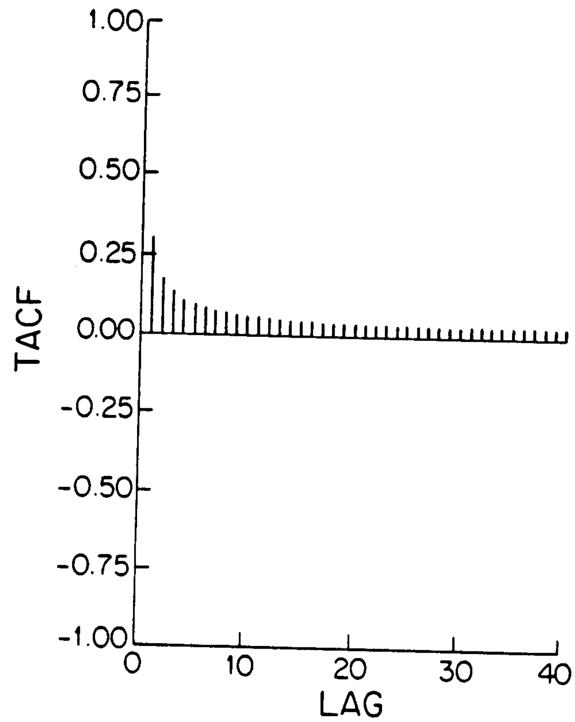

Figure 10.4.6. Theoretical ACF of the Gota River for the FGN model using K.

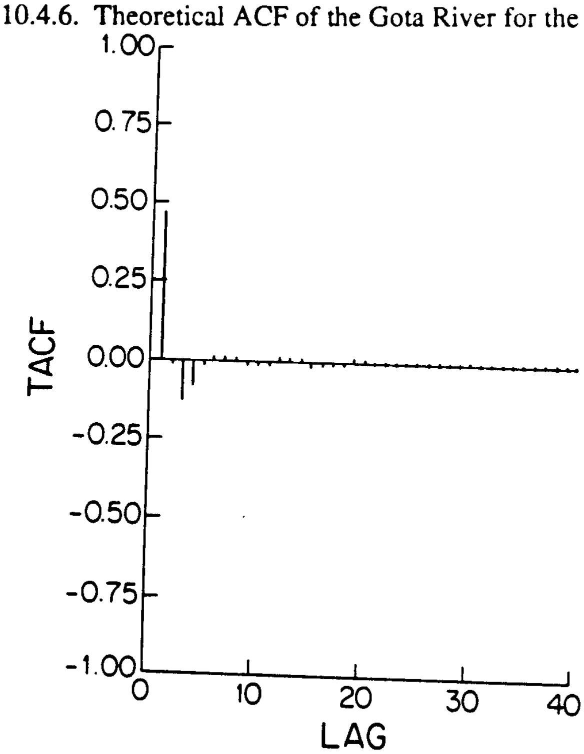

Figure 10.4.7. Theoretical ACF of the Gota River for the ARMA(2,0) model.

identification stage, the general structure of the data is determined by observing the shape of the ACF and other graphs described in Section 5.3. An appropriate number of AR and MA parameters are selected in order that the selected ARMA model fits the data as closely as possible using a minimum number of parameters. Rigorous checks are performed to insure that the white noise component of the model is not correlated. If all the modelling assumptions are satisfied, this guarantees that important historical statistics such as the ACF, the RAR, and K will be preserved reasonably well by the model.

10.5 SIMULATION STUDIES

10.5.1 Introduction

When studying statistics such as the RAR and K, information is required regarding first, second, and perhaps higher order moments of the statistics. In general, it would be most advantageous to know the exact distribution of the statistic under study. Three approaches are available to obtain knowledge regarding the mathematical properties of a specified statistic. One method is to *derive an exact analytical expression* for the moments and perhaps the distribution of the statistic. Except for special cases of the lower order moments of a statistic, this precise procedure is often analytically intractable. Only after extensive research, Anis and Lloyd (1976) were able to derive in [10.3.15] the exact expression for the expected value of the RAR for NID summands.

A second approach is to *develop asymptotic formulae* for the distributional properties of a given statistic. This approximate procedure may yield results that are useful in certain situations, while in other circumstances the output may suffer from lack of accuracy, especially for small N. Feller (1951), for example, proved an asymptotic relationship that is valid for the expected value of the adjusted range and also the RAR of IID random variables (see [10.3.5] and [10.3.6], respectively). Siddiqui (1976) derived asymptotic expressions for calculating the expected value of the RAR for any short memory process.

In the third approach, *simulation* is used to determine as accurately as desired the distributional attributes of a given statistic. In Section 10.6, Monte Carlo procedures are utilized to obtain the empirical distribution of the RAR and K. Although some researchers may argue that simulation may be relatively costly with respect to computer usage, the fact of the matter is that answers are needed now to help solve present day engineering problems. In addition, because of the vast mathematical complexity that is often required to prove exact analytical solutions, simulation results may help to economize academic endeavours by delineating the more promising avenues of research that could also be scrutinized analytically. Finally, it should be borne in mind that in comparison with an exact analytical solution, simulation provides a straightforward but equally correct resolution to the problem of the distributional characteristics of a particular statistic. The theory and practice of simulating with ARMA models are discussed in detail in Chapter 9 while an exact simulation method for use with FGN is given in this chapter in Section 10.4.6.

The simulation investigations of this section deal primarily with the estimated mean and variance of a certain statistic. Suppose that $\bar{N}$ independent simulations of a time series $z_1, z_2, \ldots, z_N$, are obtained and that a statistic $T = T(z_1, z_2, \ldots, z_N)$ is calculated in each simulated series. The empirical *mean* of T is then given by

$$\bar{T} = \frac{1}{\bar{N}}\sum_{i=1}^{\bar{N}} T_i \qquad [10.5.1]$$

where T_i is the value of T in the ith simulation. If each successive realization of the sequence $z_1, z_2, \ldots, z_N$, is independent of previous realizations so that the T_i are statistically independent, then the *variance* of T can be estimated by

$$V_T = \frac{1}{\bar{N}-1}\sum_{i=1}^{\bar{N}}(T_i - \bar{T})^2 \qquad [10.5.2]$$

By the central limit theorem, $\bar{T}$ will be distributed very nearly normally with mean equal to $E(\bar{T})$ and with variance approximately equal to $V_T/\bar{N}$. Thus, the standard deviation and confidence intervals of the expected value being estimated are readily obtained.

If $\bar{N}$ white noise series of length N are being simulated, then it is correct to simulate a single time series of length $\bar{N}N$ and then subdivide it into $\bar{N}$ series with N values in each series. However, if a correlated series is being simulated, the aforementioned procedure should not be followed. For instance, if $\bar{N}$ FGN series with $0.5 < H < 1$ are being formulated by first generating a long series of length $\bar{N}N$ and then subdividing this into $\bar{N}$ subsequences of length N, then the resulting T_i will in general be correlated. Therefore, the resulting estimate for $E(T)$ in [10.5.1] will be less precise (i.e., have larger variance), and the estimate of the variance of T in [10.5.2] will be underestimated, so that correct standard deviations and confidence intervals for $E(T)$ will not be available.

10.5.2 Simulation of Independent Summands

The Rescaled Adjusted Range

Mandelbrot and Wallis (1969e) reported simulation experiments which indicated that the expected value of the RAR for IID summands is virtually independent of the underlying distribution. However, as was pointed out by Taqqu (1970), the simulation study of Mandelbrot and Wallis (1969e) contained a serious programming error in the calculation of the RAR. Accordingly, another study of the robustness of the expected value of the RAR with respect to the underlying distribution is required.

A simulation study is performed for various types of white noise series varying in length from $N = 5$ to $N = 200$. For each value of N, the number of series of length N that are generated is $\bar{N} = 10{,}000$. The expected values of the RAR are determined by using [10.5.1] for the following independent summands: (1) normal, (2) gamma with shape parameter 0.1, (3) symmetric stable with characteristic exponent $\alpha = 1.3$, and (4) Cauchy. The simulation results for $E(\bar{R}^*_N)$ at specific values of N for the aforementioned summands are listed in Table 10.5.1. The standard deviations of the estimated values of $E(\bar{R}^*_N)$ are determined by using the square root of [10.5.2] and are given in parentheses below the estimates in Table 10.5.1. The exact values of $E(\bar{R}^*_N)$ for NID random variables are calculated by using the formula of Anis and Lloyd (1976) that is written in [10.3.15]. Comparisons of columns 2 and 4 to 7 reveal that the expected value of the RAR is indeed rather insensitive to the underlying distribution for the values of N that are considered. Even for Cauchy summands, the expected value and variance of the RAR are quite similar to the NID case. The asymptotic results of Feller (1951) for $E(\bar{R}^*_N)$ of IID summands are determined by using [10.3.6] and are tabulated in Table 10.5.1. A perusal of the asymptotic and other entries in the table discloses that the approximation given by Feller's results improves with increasing N.

Anis and Lloyd (1975) developed analytical formulae for the expected value of the crude and adjusted ranges of independent gamma random variables. For highly skewed independent gamma summands, the local Hurst coefficient for the crude and adjusted ranges possessed values greater than 0.5 for N less than 1000. However, the results of Table 10.5.1 indicate that the expected values of the RAR for IID summands are approximately independent of the underlying distribution even if that distribution is gamma. Therefore, as was confirmed by O'Connell (1976), Anis and Lloyd's (1975) results do not hold for the RAR. In addition, Hurst studied K for the RAR and not the Anis and Lloyd local Hurst coefficient for the crude and adjusted ranges.

The Hurst Coefficient

As was mentioned previously, the Hurst statistic of primary import is the RAR. Nevertheless, because the Hurst coefficient K has been extensively investigated during the past quarter of a century, this fact may insure the survival of K as an important hydrological statistic for some time to come. Therefore, some statistical properties of K and other exponents are investigated.

First, it should be noted that because of the research results of Anis and Lloyd (1976) in [10.3.15], K can be evaluated analytically for NID summands. Let K' be the Hurst coefficient calculated by using

$$K' = \log E(\bar{R}^*_N)/(\log N - \log 2) \qquad [10.5.3]$$

where $E(\bar{R}^*_N)$ is determined exactly by using [10.3.15]. It follows from Jensen's inequality (Rao, 1973, p. 57) that for finite N,

$$E(K) < K' \qquad [10.5.4]$$

In Table 10.5.2, the magnitudes of K' from [10.5.3] are listed for the length of series N ranging from 5 to 200. When 10,000 series are generated for NID random variables for each N, then the expected value of K can be estimated by utilizing [10.5.1], while the standard deviation of $E(K)$ can be calculated by using the square root of [10.5.2]. In Table 10.5.2, the estimated values of $E(K)$ for various time series lengths are catalogued. The standard deviation of each estimate is contained in the parentheses below the estimate. A comparison of columns 2 and 3 in Table 10.5.2 demonstrates that the inequality in [10.5.4] is valid. However, the difference between $E(K)$ and K' is negligible. Therefore, [10.5.3] provides a viable means for estimating the expected value of K for NID summands. In addition, the Hurst coefficient K is obviously a function of the sample size, and for increasing N the coefficient K attenuates toward its asymptotic value of 0.5. However, for small and moderate values of N, the statistic K is significantly larger than 0.5.

The coefficient K constitutes one method of estimating the generalized Hurst coefficient h in [10.3.2]. Another approach is to evaluate h by using the estimate YH of Gomide (1975) that is given in [10.3.7]. By taking logarithms of [10.3.7], an explicit expression for YH is

Table 10.5.1. Expected values of the RAR for some IID summands.

N	Analytical Results		Simulation Results*			
	Anis and Lloyd (1976)	Feller (1951)	Normal	Gamma	Stable	Cauchy
5	1.9274	2.8025	1.9273 (0.0027)	1.9851 (0.0018)	1.9264 (0.0022)	1.9506 (0.0026)
10	3.0233	3.9633	3.0302 (0.0060)	3.0330 (0.0039)	2.9699 (0.0047)	3.0556 (0.0056)
15	3.8812	4.8541	3.8826 (0.0084)	3.8356 (0.0056)	3.7571 (0.0064)	3.8987 (0.0079)
20	4.6111	5.6050	4.6047 (0.0100)	4.5141 (0.0071)	4.4408 (0.0075)	4.6214 (0.0098)
25	5.2576	6.2666	5.2540 (0.0116)	5.1213 (0.0085)	5.0044 (0.0088)	5.2889 (0.0115)
30	5.8443	6.8647	5.8770 (0.0131)	5.6709 (0.0097)	5.5681 (0.0098)	5.8767 (0.0130)
35	6.3851	7.4147	6.4214 (0.0145)	6.1707 (0.0109)	6.0090 (0.0106)	6.3974 (0.0143)
40	6.8895	7.9267	6.8920 (0.0158)	6.6605 (0.0121)	6.5037 (0.0118)	6.9075 (0.0155)
45	7.3640	8.4075	7.3595 (0.0169)	7.0938 (0.0132)	6.9010 (0.0125)	7.3934 (0.0166)
50	7.8133	8.8623	7.7785 (0.0180)	7.5012 (0.0141)	7.3184 (0.0132)	7.8540 (0.0178)
60	8.6502	9.7081	8.6246 (0.0198)	8.3061 (0.0159)	8.0670 (0.0148)	8.6263 (0.0195)
70	9.4210	10.4860	9.4453 (0.0215)	9.0632 (0.0178)	8.7242 (0.0158)	9.4454 (0.0211)
80	10.1392	11.2100	10.1349 (0.0233)	9.7327 (0.0194)	9.3732 (0.0172)	10.1336 (0.0232)
90	10.8143	11.8900	10.8208 (0.0248)	10.4068 (0.0209)	9.9544 (0.0183)	10.8857 (0.0248)
100	11.4533	12.5331	11.4775 (0.0262)	10.9769 (0.0224)	10.5593 (0.0196)	11.4546 (0.0258)
125	12.9243	14.0125	12.9617 (0.0299)	12.4280 (0.0255)	11.8353 (0.0220)	12.9619 (0.0292)
150	14.2556	15.3499	14.1956 (0.0323)	13.6864 (0.0285)	13.0622 (0.0240)	14.2636 (0.0323)
175	15.4806	16.5798	15.4198 (0.0349)	14.8752 (0.0315)	14.1069 (0.0261)	15.4971 (0.0354)
200	16.6214	17.7245	16.5938 (0.0376)	15.9992 (0.0337)	15.1381 (0.0281)	16.6259 (0.0376)

*The parenthetical values are standard deviations

Table 10.5.2. Hurst coefficients for NID summands.

N	K'	$E(K)$*	YH'
5	0.7161	0.7032 (0.0016)	0.3375
10	0.6874	0.6750 (0.0013)	0.4315
15	0.6731	0.6629 (0.0011)	0.4591
20	0.6638	0.6540 (0.0010)	0.4725
25	0.6571	0.6469 (0.0009)	0.4805
30	0.6519	0.6420 (0.0008)	0.4859
35	0.6477	0.6385 (0.0008)	0.4897
40	0.6442	0.6365 (0.0007)	0.4926
45	0.6413	0.6335 (0.0007)	0.4948
50	0.6387	0.6305 (0.0007)	0.4967
60	0.6344	0.6270 (0.0007)	0.4994
70	0.6309	0.6235 (0.0006)	0.5014
80	0.6279	0.6213 (0.0006)	0.5029
90	0.6254	0.6186 (0.0006)	0.5040
100	0.6233	0.5156 (0.0006)	0.5049
125	0.6189	0.6129 (0.0005)	0.5066
150	0.6154	0.6100 (0.0005)	0.5078
175	0.6127	0.6070 (0.0005)	0.5086
200	0.6103	0.6051 (0.0005)	0.5092

*The parenthetical values are standard deviations.

$$YH = (\log\bar{R}^*_N - \log 1.2533)/\log N \quad [10.5.5]$$

Although the expected value of YH could be determined from simulation experiments, an alternative analytical procedure is to substitute $E(\bar{R}^*_N)$ from [10.3.15] for $\bar{R}^*_N$ in [10.5.5] and then to estimate YH by YH' by using [10.5.5]. In Table 10.5.2, the values of YH' are tabulated for different time series lengths. It is obvious that YH' is a function of the sample size and that YH' provides a closer approximation to the limiting value of 0.5 than does K.

10.5.3 Simulation of Correlated Summands

Long Memory Models

By utilizing [10.4.30], it is possible to simulate exactly synthetic traces of FGN. Because only short memory approximations to FGN processes were previously available for simulation purposes, the exact method should prove useful for checking former FGN simulation studies and also for exploring new avenues of research for long memory models. Of particular importance are Monte Carlo studies to investigate the statistical properties of FGN processes. Consider, for example, the behaviour of the RAR for FGN models. For time series varying in length from $N=5$ to $N=200$ a total of 10,000 simulated sequences are generated for each value of N. Because the RAR statistic is not a function of the mean and variance of a FGN process, it is convenient to assign the mean a value of zero and the variance a magnitude of 1 when performing the simulations using [10.4.30]. By utilizing [10.5.1] and [10.5.2], the expected values of the RAR and variances, respectively, are calculated. Table 10.5.3 records the estimates of $E(\bar{R}^*_N)$ and the corresponding standard deviations in brackets for FGN models with $H=0.7$ and 0.9. From an inspection of the entries in Table 10.5.3, it is obvious that $E(\bar{R}^*_N)$ increases in magnitude for larger N. Furthermore, at a given value of N the expected value of the RAR is greater for a FGN model with $H=0.9$ than it is a for a FGN process with $H=0.7$.

Short Memory Models

In Chapter 9, improved procedures are given for generating synthetic traces using ARMA models. In particular, the WASIM1 (see Section 9.3) and WASIM2 (Section 9.4) procedures are recommended for use with ARMA models. When either WASIM1 or WASIM2 is employed, random realizations of the process under consideration are used as starting values. Since fixed initial values are not utilized, systematic bias is avoided in the generated data.

As a typical example of a short memory process, consider the Markov model of Section 3.2.1 given by

$$z_t = \phi_1 z_{t-1} + a_t \quad [10.5.6]$$

where t equals $1,2,\ldots,N$, ϕ_1 is the AR parameter, and a_t is the white noise that is NID$(0,\sigma_a^2)$. By using WASIM2, a total of 10,000 synthetic sequences are generated for specific values of N for Markov processes with $\phi_1 = 0.3$, 0.5 and 0.7. Because the RAR is independent of the variance of the innovations, a value such as unity may be used for σ_a^2 in the simulation study. In Tables 10.5.4 to 10.5.6, the expected values of the RAR and corresponding standard deviations in parentheses are given for the three Markov models. Comparisons of the third columns in these tables reveal that the expected value of the RAR increases for increasing N and ϕ_1.

Table 10.5.3. Expected values of the RAR for FGN models.

N	FGN Models*	
	H=0.7	H=0.9
5	1.9682	2.0100
	(0.0026)	(0.0025)
10	3.2716	3.5031
	(0.0062)	(0.0061)
15	4.3946	4.8751
	(0.0091)	(0.0094)
20	5.3972	6.1579
	(0.0116)	(0.0125)
25	6.3351	7.4051
	(0.0141)	(0.0155)
30	7.2066	8.6032
	(0.0165)	(0.0187)
35	8.0515	9.7839
	(0.0188)	(0.0216)
40	8.8767	10.9431
	(0.0205)	(0.0241)
45	9.6650	12.0926
	(0.0227)	(0.0271)
50	10.4007	13.2284
	(0.0247)	(0.0298)
60	11.8233	15.3575
	(0.0280)	(0.0352)
70	13.2003	17.4965
	(0.0322)	(0.0413)
80	14.5205	19.5945
	(0.0356)	(0.0461)
90	15.7709	21.6075
	(0.0389)	(0.0518)
100	16.9241	23.5818
	(0.0420)	(0.0573)
125	19.8877	28.5197
	(0.0494)	(0.0700)
150	22.6178	33.2646
	(0.0571)	(0.0831)
175	25.2291	38.0410
	(0.0638)	(0.0964)
200	27.7601	42.6710
	(0.0701)	(0.1080)

*The parenthetical values are standard deviations.

Table 10.5.4. Expected values of the RAR for a Markov model with $\phi_1 = 0.3$.

N	$E(\bar{R}^*_N)$	
	Asymptotic	Simulated*
5	3.8192	1.9875 (0.0026)
10	5.4011	3.3410 (0.0062)
15	6.6150	4.4633 (0.0089)
20	7.6383	5.4261 (0.0114)
25	8.5390	6.2853 (0.0135)
30	9.3550	7.0666 (0.0156)
35	10.1045	7.7976 (0.0175)
40	10.8022	8.5022 (0.0188)
45	11.4575	9.1493 (0.0205)
50	12.0772	9.7347 (0.0221)
60	13.2299	10.8709 (0.0242)
70	14.2900	11.9207 (0.0273)
80	15.2766	12.9177 (0.0296)
90	16.2033	13.8181 (0.0317)
100	17.0798	14.6243 (0.0335)
125	19.0958	16.6970 (0.0380)
150	20.9184	18.5288 (0.0424)
175	22.5944	20.1758 (0.0459)
200	24.1545	21.7339 (0.0491)

*The parenthetical values are standard deviations.

Table 10.5.5. Expected values of the RAR for a Markov model with $\phi_1 = 0.5$.

N	$E(\bar{R}^*_N)$	
	Asymptotic	Simulated*
5	4.8541	2.0194 (0.0025)
10	6.8647	3.5438 (0.0061)
15	8.4075	4.8738 (0.0092)
20	9.7081	6.0432 (0.0120)
25	10.8540	7.1131 (0.0147)
30	11.8900	8.0779 (0.0171)
35	12.8426	8.9858 (0.0194)
40	13.7294	9.8655 (0.0212)
45	14.5622	10.6837 (0.0233)
50	15.3499	11.4170 (0.0252)
60	16.8150	12.8455 (0.0283)
70	18.1622	14.1721 (0.0320)
80	19.4163	15.4320 (0.0350)
90	20.5941	16.5726 (0.0376)
100	21.7080	17.5991 (0.0400)
125	24.2703	20.2124 (0.0459)
150	26.5868	22.5342 (0.0515)
175	28.7170	24.6356 (0.0562)
200	30.6998	26.6039 (0.0603)

*The parenthetical values are standard deviations.

Table 10.5.6. Expected values of the RAR for a Markov model with $\phi_1 = 0.7$.

N	$E(\bar{R}^*_N)$	
	Asymptotic	Simulated*
5	6.6713	2.0435 (0.0025)
10	9.4346	3.7235 (0.0059)
15	11.5550	5.2915 (0.0091)
20	13.3425	6.7304 (0.0123)
25	14.9174	8.0874 (0.0154)
30	16.3412	9.3309 (0.0184)
35	17.6505	10.5117 (0.0212)
40	18.8692	11.6603 (0.0235)
45	20.0138	12.7462 (0.0262)
50	21.0964	13.7239 (0.0286)
60	23.1100	15.6339 (0.0331)
70	24.9616	17.4191 (0.0378)
80	26.6851	19.1225 (0.0419)
90	28.3038	20.6666 (0.0454)
100	29.8348	22.0685 (0.0490)
125	33.3564	25.6001 (0.0570)
150	36.5401	28.7578 (0.0648)
175	39.4678	31.6509 (0.0716)
200	42.1928	34.3412 (0.0772)

*The parenthetical values are standard deviations.

It is also possible to compare the estimated expected value of the RAR for a Markov model to an analytical large-sample approximation that is given by Siddiqui (1976) as

$$E(\bar{R}^*_N) = \{(\pi N/2)[(1-\phi_1^2)/(1-\phi_1)^2]\}^{1/2} \qquad [10.5.7]$$

In Tables 10.5.4 to 10.5.6, the output from [10.5.7] for the three Markov models are catalogued. A perusal of these tables demonstrates that Siddiqui's approximation for $E(\bar{R}^*_N)$ is not too accurate for the cases considered, and the precision decreases for increasing ϕ_1.

10.6 PRESERVATION OF THE RESCALED ADJUSTED RANGE

10.6.1 Introduction

A major challenge in stochastic hydrology is to determine models that preserve important historical statistics such as the rescaled adjusted range (RAR), or equivalently the Hurst coefficient K. The major finding of this section is that ARMA models do statistically preserve the historical RAR statistics or equivalently the Hurst coefficients denoted using K's. This interesting scientific result is what solves the riddle of the Hurst Phenomenon.

After fitting ARMA models to 23 annual geophysical time series, simulation studies are carried out to determine the small sample *empirical cumulative distribution function (ECDF)* of the RAR or K for various ARMA models. The ECDF for each of these statistics is shown to be a function of the time series length N and the parameter values of the specific ARMA process being considered. Furthermore, it is possible to determine as accurately as desired the distribution of the RAR or K. A theorem is given to obtain confidence intervals for the ECDF in order to guarantee a prescribed precision. Then it is shown by utilizing simulation results and a given statistical test that ARMA models do preserve the observed RAR or K of the 23 geophysical time series.

10.6.2 ARMA Modelling of Geophysical Phenomena

In this section, ARMA models are determined for 23 yearly geophysical time series. Table 10.6.1 lists the average annual riverflows and miscellaneous geophysical phenomena that are modelled. The riverflows are the longer records that are available in a paper by Yevjevich (1963). Although the flows were converted to cubic meters per second, it is irrelevant which units of measurement are used, since the AR and MA parameter estimates for the ARMA models fitted to the data are independent of the measuring system used. The mud varve, temperature, rainfall, sunspot numbers, and minimum flows of the Nile River are obtained from articles by De Geer (1940), Manley (1953, pp. 255-260), Kendall and Stuart (1963, p. 343), Waldmeier (1961), and Toussoun (1925), respectively.

Table 10.6.2 lists 12 sets of tree ring indices comprising six different species of trees from western North America. The indices labelled Snake are from a book by Schulman's (1956, p. 77), and the rest were selected from a report by Stokes et al. (1973).

By employing the three stages of model construction presented in detail in Part III, the most appropriate ARMA model from [3.4.32] is fitted to each of the 23 time series. Table 10.6.3 catalogues the type of ARMA model, Box-Cox transformation from [3.4.30], parameter estimates and standard errors (SE's) for each data set. The SE's are given in parentheses. For all the Box-Cox transformations, the constant is set equal to zero. When $\lambda = 1$ there is no

Table 10.6.1. Annual riverflows and miscellaneous geophysical data.

Code Name	Type	Location	Period	Length N
Mstouis	Mississippi River	St. Louis, Missouri	1861-1957	96
Neumunas	Neumunas River	Smalininkai, USSR	1811-1943	132
Danube	Danube River	Orshava, Romania	1837-1957	120
Rhine	Rhine River	Basle, Switzerland	1807-1957	150
Ogden	St. Lawrence River	Ogdensburg, New York	1860-1957	97
Gota	Gota River	Sjotorp-Vanersburg, Sweden	1807-1957	150
Espanola	mud varves	Espanola, Ontario	-471 to -820	350
Temp	temperature data	English Midlands	(Swedish time)	
Precip	precipitation	London, England	1698-1952	255
Sunyr	yearly sunspots	sun	1813-1912	100
Minimum	minimum flows of	Rhoda, Egypt	1798-1970	163
	Nile River		622-1469	848

Table 10.6.2. Tree ring indicies data.

Code Name	Type of Tree	Location	Period	Length N
Snake	Douglas fir	Snake River Basin	1282-1950	669
Exshaw	Douglas fir	Exshaw, Alberta, Canada	1460-1965	506
Naramata	Ponderosa pine	Naramata, B.C., Canada	1451-1965	515
Dell	Limber pine	Dell, Montana	1311-1965	655
Lakeview	Ponderosa pine	Lakeview, Oregon	1421-1964	544
Ninemile	Douglas fir	Nine Mile Canyon, Utah	1194-1964	771
Eaglecol	Douglas fir	Eagle, Colorado	1107-1964	858
Navajo	Douglas fir	Navajo National Monument (Belatakin), Arizona	1263-1962	700
Bryce	Ponderosa pine	Bryce Water Canyon, Utah	1340-1964	625
Tioga	Jeffrey pine	Tioga Pass, California	1304-1964	661
Bigcone	Big cone spruce	Southern California	1458-1966	509
Whitemtn	Bristlecone pine	White Mountains, California	800-1963	1164

transformation, while $\lambda = 0$ means that natural logarithms are taken of the data. Whenever a MLE of λ is calculated, the SE is included in parentheses.

10.6.3 Distribution of the RAR or K

Suppose the determination of the exact distribution of the RAR (i.e., $\bar{R}^*_N$) or K is required. The expected value of $\bar{R}^*_N$ is now known theoretically for both an independent and a symmetrically correlated Gaussian process (Anis and Lloyd, 1976). At present, the cumulative

Table 10.6.3. ARMA models fitted to the geophysical data.

Code Name	Model	λ*	Parameter	Value*	Parameter	Value*	Parameter	Value*
Mstouis	(0,1)	1.0	θ_1	-0.309				
				(0.094)				
Neumunas	(0,1)	0.0	θ_1	-0.222				
				(0.086)				
Danube	(0,0)	1.0						
Rhine	(0,0)	1.0						
Ogden	(3,0)	1.0	ϕ_1	0.626	ϕ_2	0.0	ϕ_3	0.184
				(0.083)				(0.086)
Gota	(2,0)	1.0	ϕ_1	0.591	ϕ_2	-0.274		
				(0.079)		(0.086)		
Espanola	(1,1)	0.0	ϕ_1	0.963	θ_1	0.537		
				(0.016)		(0.051)		
Temp	(0,2)	1.0	θ_1	-0.115	θ_2	-0.202		
				(0.063)		(0.057)		
Precip	(0,0)	0.0						
Sunyr	(9,0)	1.0	ϕ_1	1.219	ϕ_2	-0.508	ϕ_9	0.232
				(0.060)		(0.056)		(0.029)
Minimum	(2,1)	-0.778	ϕ_1	1.254	ϕ_2	-0.279	θ_1	0.842
		(0.316)		(0.060)		(0.051)		(0.049)
Snake	(3,0)	1.0	ϕ_1	0.352	ϕ_2	0.093	ϕ_3	0.100
				(0.039)		(0.041)		(0.039)
Exshaw	(1,1)	1.0	ϕ_1	0.725	θ_1	0.395		
				(0.067)		(0.090)		
Naramata	(2,0)	1.0	ϕ_1	0.196	ϕ_2	0.131		
				(0.044)		(0.044)		
Dell	(2,0)	1.0	ϕ_1	0.367	ϕ_2	0.185		
				(0.039)		(0.039)		
Lakeview	(3,0)	0.717	ϕ_1	0.525	ϕ_2	0.0	ϕ_3	0.143
		(0.130)		(0.038)				(0.039)
Ninemile	(2,1)	0.684	ϕ_1	1.225	ϕ_2	-0.274	θ_1	0.850
		(0.060)		(0.063)		(0.047)		(0.049)
Eaglecol	(2,1)	0.624	ϕ_1	1.156	ϕ_2	-0.237	θ_1	0.693
		(0.054)		(0.114)		(0.082)		(0.103)
Navajo	(1,1)	1.0	ϕ_1	0.683	θ_1	0.424		
				(0.082)		(0.103)		
Bryce	(1,0)	1.366	ϕ_1	0.598				
		(0.107)		(0.033)				
Tioga	(1,0)	1.458	ϕ_1	0.556				
		(0.098)		(0.033)				
Bigcone	(2,0)	1.0	ϕ_1	0.375	ϕ_2	0.159		
				(0.044)		(0.044)		
Whitemtn	(1,1)	1.414	ϕ_1	0.641	θ_1	0.408		
		(0.061)		(0.086)		(0.104)		

*The parenthetical values are standard errors (SE's).

distribution function (CDF) of $\bar{R}^*_N$ for a white noise process and in general any ARMA model is analytically intractable. However, by simulation it is possible to determine as accurately as is desired for practical purposes the CDF for $\bar{R}^*_N$. Because both $\bar{R}^*_N$ and K are functions of N, their CDF's are defined for a particular length of series N. The CDF for $\bar{R}^*_N$ is

$$F = F(r;N,\phi,\theta) = Pr(\bar{R}^*_N \le r) \qquad [10.6.1]$$

where N is the length of each individual time series, ϕ is the set of known AR parameters, θ is the set of known MA parameters, and r is any possible value of $\bar{R}^*_N$.

When simulating a time series of length N, it is recommended to employ the improved simulation techniques of Chapter 9. In this section, WASIM1 from Section 9.3 is utilized for the ARMA $(0,q)$ models, while WASIM2 from Section 9.4 is used with the ARMA$(p,0)$ and ARMA(p,q) processes. Because the RAR or K is independent of the variance of the innovations, any value of σ_a^2 may be used. Consequently, it is simplest to set $\sigma_a^2 = 1$ and hence to assume that the residuals are NID(0,1).

Suppose that $\bar{N}$ simulations of length N are generated for a specific ARMA model and the $\bar{N}$ RAR's given by $\bar{R}^*_{N1}, \bar{R}^*_{N2}, \ldots, \bar{R}^*_{N\bar{N}}$, are calculated for the $\bar{N}$ simulated series, respectively. If the sample of RAR is reordered such that $\bar{R}^*_{N(1)} \le \bar{R}^*_{N(2)} \le \cdots \le \bar{R}^*_{N(\bar{N})}$, it is known that the MLE of F is given by the ECDF (Gnedenko, 1968, pp. 444-451):

$$F_{\bar{N}} = F_{\bar{N}}(r;N,\phi,\theta) = 0\,, \quad r \le \bar{R}^*_{N(1)}$$

$$F_{\bar{N}} = F_{\bar{N}}(r;N,\phi,\theta) = k/N\,, \quad \bar{R}^*_{N(k)} < r \le \bar{R}^*_{N(k+1)} \qquad [10.6.2]$$

$$F_{\bar{N}} = F_{\bar{N}}(r;N,\phi,\theta) = 1\,, \quad r > \bar{R}^*_{N(\bar{N})}$$

The Kolmogorov theorem (Gnedenko, 1968, p. 450) can be used to obtain confidence intervals for $F_{\bar{N}}$ and to indicate the number of samples $\bar{N}$ necessary to guarantee a prescribed accuracy. This theorem states that if $\bar{N}$ is moderately large (it has been shown that $\bar{N} > 100$ is adequate), then

$$Pr(\max_r |F_{\bar{N}} - F| < \varepsilon/N^{1/2}) \approx K(\varepsilon) \qquad [10.6.3]$$

where

$$K(\varepsilon) = 0\,, \quad \varepsilon \le 0$$

$$K(\varepsilon) = \sum_{k=-\infty}^{\infty} (-1)^k e^{-2k^2\varepsilon^2}\,, \quad \varepsilon > 0$$

For example, when $\varepsilon = 1.63$, then $K(\varepsilon) = 0.99$. If $\bar{N} = 10^4$ simulations are done for a series of length N, then by Kolmogorov's theorem, all the values of $F_{\bar{N}}$ are accurate to at least within 0.0163 with a probability of 0.99.

In actual simulation studies, it is useful to examine the convergence of $F_{\bar{N}}$ by printing out a summary of the ECDF for increasing values of $\bar{N}$ (such as $\bar{N} = 100,200,500,1000,2000,\cdots$) until sufficient accuracy has been obtained. To curtail the computer time required in simulations, there are efficient algorithms available called "quicksorts" (Knuth, 1973) for ordering the sample values for the RAR.

If simulation studies are done for $\bar{R}^*_N$, the ECDF for K can be obtained from the transformation

$$K = \log\bar{R}^*_N/\log(N/2) \quad [10.6.4]$$

Alternatively, when the ECDF for K is known, the ECDF for $\bar{R}^*_N$ can be calculated by substituting each value of K into

$$\bar{R}^*_N = (N/2)^K \quad [10.6.5]$$

Representative ECDF's are given in Appendix A10.1 for the simulations carried out in this section. In Table A10.1.1, the ECDF of K is shown for various values of N for white noise that is NID$(0,\sigma_a^2)$. For each value of N (i.e., each row) in that table, an ECDF is determined using $\bar{N} = 10^4$ samples of length N. By substituting all values for K in this table into [10.6.5] the ECDF for the RAR can be found for each value of N.

When a particular time series is modelled by an ARMA model other than white noise, the ECDF for either $\bar{R}^*_N$ or K can be calculated by simulation for each desired value of N. Table A10.1.2 lists the ECDF of K for different values of N for a Markov process in [3.2.1] with $\phi_1 = 0.4$. By utilizing the transformation in [10.6.5] for each entry in this table, the ECDF's for $\bar{R}^*_N$ for the Markov model can be found and are shown in Table A10.1.3. Because of the transformation in [10.6.5], it is sufficient to simply have a table for either K or $\bar{R}^*_N$.

The tables of various ECDF's for different types of ARMA models are listed in the appendix of the microfiche version of the paper by Hipel and McLeod (1978a). In particular, results are given for white noise as well as Markov models with $\phi_1 = 0.1, 0.2, \ldots, 0.9$. In all of the tables, for a particular value of N the number of samples $\bar{N}$ simulated is 10^4.

For a particular ARMA model, the ECDF can be used to make inferences about $\bar{R}^*_N$ or equivalently K. For instance, the 95% confidence interval for $\bar{R}^*_N$ with $N = 100$ for a Markov or AR(1) model with $\phi_1 = 0.4$ can be determined by utilizing Table A10.1.3. Opposite $N = 100$, select the values of $\bar{R}^*_N$ below the 0.025 and 0.975 quantiles. The 95% confidence interval for the RAR is then 9.85 - 24.02. By substituting these interval limits into [10.6.4], the 95% confidence interval for K is 0.585 - 0.813. This confidence interval for K is also confirmed by referring to the appropriate entries of Table A10.1.2 opposite $N = 100$.

The ECDF tables illustrate certain properties of the RAR or K. For example, an examination of the median for K for white noise below the 0.500 quantile in Table A10.1.1, definitely shows that K slowly decreases asymptotically toward 0.5 with increasing N and is consequently a function of N. Because of this, a separate ECDF must be developed for each value of N for a specified process. Note that the median values for K in Table A10.1.1 are almost identical with the values of K tabulated in Table 10.5.2. These latter values of K are calculated by using [10.6.4] when the exact theoretical expected values of $\bar{R}^*_N$ are found from a formula given by Anis and Lloyd (1976) and also by employing simulation techniques to estimate $E(K)$. As can be seen from a perusal of Table 10.5.2, the expected value of K is obviously a function of N and decreases in magnitude with increasing N.

It can be proven theoretically that for any ARMA process the RAR or K is a function of the time series length N and the AR and MA parameters (Hipel, 1975, Appendix B). This fact is confirmed by the ECDF for the RAR for the Markov process with $\phi_1 = 0.4$ in Tables A10.1.2 and A10.1.3. It can be seen that the median and all other values of the RAR at any quantile for all of the Markov models increase in value for increasing N. When one compares the results with other Markov models given by Hipel and McLeod (1978a, microfiche appendix), the distribution of $\bar{R}^*_N$ or K is also a function of the value of the AR parameter ϕ_1.

10.6.4 Preservation of the RAR and K by ARMA Models

By employing the ECDF of the RAR or K in conjunction with a specified statistical test it is now shown that ARMA models do preserve the historically observed Hurst statistics. Because the Hurst coefficient K is widely cited in the literature, the research results for this statistic are described. However, K and $\bar{R}^*_N$ are connected by the simple transformation given in [10.6.5], and, therefore, preservation of either statistic automatically implies retention of the other by an ARMA model.

The ARMA models fitted to 23 geophysical time series ranging in length from $N = 96$ to $N = 1164$ are listed in Table 10.6.3. For exactly the same time series length N as the historical data, 10^4 simulations are done for each model to determine the ECDF of K, or equivalently $\bar{R}^*_N$. The probability p_i of having K for the ith model greater than the K calculated for the ith historical series is determined from the ith ECDF as

$$Pr(K > K_i^{obs} | \text{model}) = p_i \qquad [10.6.6]$$

where K_i^{obs} is the K value calculated for the ith observed historical time series. If the chosen ARMA model is correct, then, by definition, p_i would be uniformly distributed on (0,1). For k time series it can be shown (Fisher, 1970, p. 99) that

$$-2\sum_{i=1}^{k} \ln p_i \sim \chi^2_{2k} \qquad [10.6.7]$$

Significance testing can be done by using [10.6.7] to determine whether the observed Hurst coefficient or the RAR is preserved by ARMA models. The test could fail if the incorrect model were fitted to the data (for example, if the Ogden data were incorrectly modelled by an AR(1) process with $\phi_1 = 0.4$) or if ARMA models do not retain the Hurst K. Careful model selection was done, thereby largely eliminating the former reason for test failure. If it is thought (as was suggested by Mandelbrot and Wallis (1968)) that the observed K is larger than that implied by an appropriate Brownian domain model, then a one tailed rather than a two tailed test may be performed.

The results of the χ^2 test in [10.6.7] for the 23 geophysical phenomena confirm that there is no evidence that the observed K's, or equivalently the RAR's, are not adequately preserved by the fitted ARMA models. Table 10.6.4 summarizes the information used in the test. The observed Hurst coefficient, $E(K)$ from the simulations and the p_i value are listed for each of the time series. In Table 10.6.5, it can be seen that the calculated χ^2 value from [10.6.7] is not significant at the 5% level of significance for the 23 time series for either a one sided or a two sided test. Therefore, on the basis of the given information, ARMA models do statistically preserve K or the RAR when considering all the time series. Furthermore, when the set of annual riverflows, miscellaneous data, and tree ring indicies are inspected individually, it can be seen from Table 10.6.5 that ARMA models preserve the historical Hurst statistics for all three cases.

Table 10.6.4. Geophysical time series calculations.

Code Names	N's	Observed K's	ARMA Model $E(K)$'s	p_i's
Mstouis	96	0.648	0.667	0.624
Neumunas	132	0.660	0.649	0.420
Danube	120	0.633	0.613	0.534
Rhine	150	0.614	0.609	0.468
Ogden	97	0.894	0.832	0.149
Gota	150	0.689	0.659	0.283
Espanola	350	0.855	0.877	0.674
Temperature	255	0.694	0.646	0.157
Precip	100	0.618	0.610	0.434
Sunspot numbers	163	0.723	0.768	0.728
Minimum	848	0.815	0.786	0.264
Snake	669	0.687	0.693	0.559
Exshaw	506	0.637	0.702	0.938
Naramata	515	0.595	0.649	0.905
Dell	655	0.687	0.694	0.569
Lakeview	544	0.706	0.729	0.709
Ninemile	771	0.740	0.726	0.378
Eaglecol	858	0.645	0.747	0.995
Navajo	700	0.653	0.670	0.660
Bruce	625	0.732	0.698	0.203
Tioga	661	0.701	0.687	0.362
Bigcone	509	0.611	0.695	0.981
Whitemtn	1164	0.695	0.648	0.095

Table 10.6.5. Results of the χ^2 test for the geophysical time series.

Data Sets	Degrees of Freedom	$-2\ln\sum p_i$
Riverflows	12	11.78
Miscellaneous	10	9.46
Tree rings	24	16.08
Total	46	37.32

In Table 10.6.4, the average of the observed K's is calculated to be 0.693 with a standard deviation of 0.076. The $E(K)$ from the simulations has an average of 0.698 with a standard deviation of 0.068. The average of the observed K is, therefore, slightly less than that for the simulated case, but this difference is not statistically different.

If the results of the RAR had been given rather than K, only columns 3 and 4 of Table 10.6.4 would be different, due to the transformation in [10.6.7]. The p_i values and the results of the χ^2 test in Table 10.6.5 would be identical. Therefore, preservation of either K or $\bar{R}^*_N$ infers retention of the other statistic by ARMA models.

10.7 ESTIMATES OF THE HURST COEFFICIENT

Different estimators are available for estimating the Hurst coefficient. The purpose of this section is to compare these estimates for the 23 annual geophysical time series given in Tables 10.6.1 and 10.6.2.

From empirical studies of approximately 690 geophysical time series, Hurst (1951, 1956) found the RAR to vary as

$$\bar{R}^*_N \propto N^h \qquad [10.7.1]$$

where h is a constant often referred to as the generalized Hurst coefficient. The above equation can be written in the general form

$$\bar{R}^*_N = aN^h \qquad [10.7.2]$$

where a is a coefficient. Hurst assumed the coefficient a to have a value of $(1/2)^h$ and then estimated h by K in [10.6.4].

Siddiqui (1976) has employed the functional central limit theorem and the theory of Brownian motion to derive many statistical formulae that may be of interest to hydrologists. Of particular importance is the asymptotic result for calculating $E(\bar{R}^*)$ for ARMA processes. This formula is given as

$$E(\bar{R}^*) \simeq a'N^{1/2} \qquad [10.7.3]$$

where

$$a' = 1.2533\gamma_0^{-1/2}\left(1-\sum_{i=1}^{q}\theta_i\right)\Big/\left(1-\sum_{i=1}^{p}\phi_i\right)$$

and γ_0 is the theoretical autocovariance function at lag 0 that is evaluated by using the algorithm in Appendix A3.2 with $\sigma_a^2 = 1$, θ_i is the ith MA parameter and ϕ_i is the ith AR parameter. If the random variables are IID, a special case of [10.7.3] that was previously derived by Feller (1951) is

$$E(\bar{R}^*_N) \simeq 1.2533N^{1/2} \qquad [10.7.4]$$

By comparing [10.7.3] and [10.7.2], a possible alternative method of evaluating h may be to employ the equation

$$\bar{R}^*_N = a'N^{SH} \qquad [10.7.5]$$

where SH is Siddiqui's estimate of the generalized Hurst coefficient h. When logarithms are taken of [10.7.5], Siddiqui's estimate for h is (Siddiqui, 1976)

$$SH = (\log\bar{R}^*_N - \log a')(\log N)^{-1} \qquad [10.7.6]$$

It should be noted that due to the way Hurst (1951, 1956) and Siddiqui (1976) calculate the coefficient a in [10.7.2], the Hurst coefficient K and the Siddiqui coefficient SH are in fact two different statistics. Nevertheless, as was suggested by Siddiqui (1976), it may be of interest to determine whether h exhibits the Hurst phenomenon if the estimate SH is employed. Accordingly, for the 23 geophysical time series given in Tables 10.6.1 and 10.6.2 the K and SH statistics are compared.

Table 10.6.3 lists the ARMA models fitted to the 23 time series. If a Box-Cox transformation is included in a model, then K and SH are calculated for the transformed series to which the model is fit. This is because the formula for calculating SH in [10.7.6] does not have the capability of incorporating a Box-Cox transformation in order to get an estimate of SH for the untransformed data. Table 10.7.1 displays the values of K and SH that are calculated for each time series by using [10.6.4] and [10.7.6], respectively. Notice that the entries for K in Table 10.7.1 differ from the K values in Table 10.6.4 wherever the data used in Table 10.7.1 have been transformed by a Box-Cox transformation.

An examination of Table 10.7.1 reveals that in all cases except three, the value of SH is less than K for the corresponding time series. The K statistic has an arithmetic mean of 0.701 with a standard deviation of 0.084. However, the mean of the SH statistic is 0.660 and possesses a standard deviation of 0.131. The mean value of SH is, therefore, well within 2 standard deviations of 0.500.

Another technique to estimate h can be found by comparing [10.7.4] and [10.7.2]. Accordingly, Gomide (1975) suggests the following equation to evaluate h:

$$\bar{R}^*_N = 1.2533N^{YH} \qquad [10.7.7]$$

where YH is Gomide's estimate of the generalized Hurst coefficient h. By taking logarithms of [10.7.7], Gomide's estimate of h is

$$YH = (\log\bar{R}^*_N - \log 1.2533)(\log N)^{-1} \qquad [10.7.8]$$

When [10.7.8] is utilized to estimate the Hurst coefficient, Gomide (1975) obtains an average value for YH of 0.57 for the 690 series considered by Hurst (1951, 1956). On the other hand, Hurst (1951, 1956) calculated K to have an average of 0.73 for the 690 series. Therefore, lower values are obtained for the Hurst coefficient h if YH is employed rather than K.

Table 10.7.1 lists the values of YH for the same 23 geophysical time series that are considered for SH. Therefore, if a Box-Cox transformation is included with an ARMA model in Table 10.6.3, then YH is determined for the transformed series to which the model is fit. Obviously, because YH, as calculated in [10.7.8], is not a function of the ARMA model parameters, it is not, in general, necessary to consider the transformed series. However, the aforementioned procedure is adopted so that appropriate comparisons can be formulated for the three estimates given in Table 10.7.1.

Table 10.7.1. Estimates of the Hurst coefficient.

Code Names	K's	SH's	YH's
Mstouis	0.648	0.451	0.500
Neumunas	0.677	0.499	0.535
Danube	0.633	0.495	0.495
Rhine	0.614	0.484	0.484
Ogden	0.894	0.436	0.709
Gota	0.689	0.504	0.549
Espanola	0.928	0.455	0.779
Temp	0.694	0.521	0.567
Precip	0.615	0.473	0.473
Sunyr	0.723	0.570	0.580
Minimum	0.817	0.462	0.699
Snake	0.687	0.475	0.579
Exshaw	0.637	0.420	0.530
Naramata	0.595	0.435	0.492
Dell	0.687	0.475	0.579
Lakeview	0.703	0.499	0.590
Ninemile	0.727	0.466	0.617
Eaglecol	0.761	0.485	0.650
Navajo	0.653	0.468	0.550
Bryce	0.734	0.513	0.620
Tioga	0.704	0.498	0.594
Bigcone	0.611	0.404	0.507
Whitment	0.695	0.530	0.595

A perusal of Table 10.7.1 shows that for each time series the values of both SH and YH is consistently less than the magnitude of K. For the series to which white noise models are fit in Table 10.6.3 (i.e., Danube, Rhine and Precip), the values of YH and SH in Table 10.7.1 are equivalent. However, for all the other data sets the magnitudes of SH are less than YH. The mean of the 23 YH values is 0.577 with a standard deviation of 0.078. The YH statistic is within one standard deviation of 0.500. Therefore, it can perhaps be argued that for the data considered, the Hurst phenomenon is not significant for the YH statistic. A similar argument can be made for the SH estimate of h.

10.8 CONCLUSIONS

The pursuit of possible explanations to solve the riddle of the Hurst phenomenon has stimulated decades of valuable research by both hydrologists and statisticians. The Hurst researchers are analogous to the inquisitive archaelogists of the 19th and early 20th centuries who sought to find the treasures of the ancient Egyptians in long forgotten temples, pyramids and tombs. Like the archaelogists, during their search the Hurst scientists have unearthed many valuable treasures that have attracted the world-wide attention of their colleagues. However, the main treasure find is the one described in Section 10.6. In that section, ARMA models are shown to preserve statistically the observed RAR and K when fitted to a variety of geophysical time series. In other words, the fitted ARMA models indirectly account for the measured Hurst statistics, which are usually significantly larger than 0.5 (see Table 10.6.4). Because important stochastic characteristics of hydrologic time series are retained by ARMA models, this should give engineers confidence in water resource projects that are designed with the aid of simulation techniques. In particular, the RAR statistic is directly related to storage problems, and this makes ARMA models desirable for reservoir design, operation, and evaluation.

Besides the main solution to the Hurst riddle given in Section 10.6, many other interesting discoveries have been made. In addition to the Hurst coefficient K defined in [10.3.4], other coefficients have been suggested to model the generalized Hurst coefficient h given in [10.3.2]. For example, Gomide (1975), Siddiqui (1976), Anis and Lloyd (1976) and Wallis and Matalas (1970) proposed alternative procedures to model h. One of the major reasons for developing alternative exponents to K was to produce a coefficient that would reach its limiting value of 0.5 more quickly than K would. Nevertheless, it must be borne in mind that the definition of the Hurst phenomenon is based on a comparison of the value of K in small and moderate sample sizes to its large sample value of 0.5. If the empirical, or theoretical, value of another estimate of h is compared for finite time series length to its asymptotic magnitude of 0.5, the Hurst phenomenon should probably be redefined in terms of that statistic. However, because of the inherent statistical properties of the RAR, it is recommended that future research primarily be devoted to the study of this statistic and that less emphasis be put on the various definitions of the Hurst coefficient. Some interesting insights into problems related to the Hurst phenomenon are provided by Klemes and Klemes (1988). Further research into the Hurst phenomenon and long-range dependence is provided by Bhattacharya et al. (1983) and Poveda and Mesa (1988) while Beran (1992) carries out a partial survey of long-range dependence research. Kunsch (1986) provides an approach for discriminating between monotonic trends and long-range dependence. Finally, Cox (1991) links non-linearity and time irreversibility with long-range dependence.

Feller (1951) proved that the asymptotic formula for the expected value of the adjusted range in [10.3.5] is valid for IID random variables. As is shown in [10.3.6] for large samples, Feller's equation is also correct for the expected value of the RAR for IID summands. The exact analytical expression for the expected value of the RAR for NID summands was derived by Anis and Lloyd (1976) and is written in [10.3.15]. For finite samples, the simulation and analytical results of Table 10.5.1 indicate that the expected values of the RAR and hence K are functions of the sample size but are virtually independent of the underlying distribution for IID summands. Accordingly, it has been suggested that the Hurst phenomenon could be explained by a combination of transcience and autocorrelation (Wallis and O'Connell, 1973). This implies that perhaps either a short memory or a long memory model that takes into account the autocorrelation structure of a time series may explain the Hurst phenomenon. Perhaps a better way to phrase this is that if a given stochastic model, that is fit to a given data set, preserves the important historical statistics such as the RAR and K, then that model may indirectly account for the Hurst phenomenon. Therefore, it can be argued that a resolution to the controversies related to the Hurst phenomenon boils down to determining stochastic models that preserve the RAR, as well as other relevant historical statistics.

If a stochastic model is to retain the historical statistical characteristics of a time series, then the model must provide a good statistical fit to the data. This can be accomplished in practice by following the identification, estimation, and diagnostic check stages of model construction described in Part III of the book. For long memory FGN processes the authors have developed an efficient estimation procedure using the method of maximum likelihood (Section 10.4.3), and a technique for calculating the model residuals so that they can be tested by appropriate diagnostic checks (Section 10.4.4). Moreover, in Section 10.4.5 a method is given for calculating one step ahead MMSE forecasts for a FGN model. Finally, in Section 10.4.6 a technique is presented for exactly simulating FGN such that the synthetic traces will lie outside the Brownian domain for the parameter H in the range $0.5 < H < 1$.

Short memory models provide an alternative approach to FGN processes for modelling hydrological time series. In particular, the ARMA family of short memory models possesses great potential for widespread applications to water resource as well as other geophysical and environmental problems. Klemes et al. (1981) maintain that given the socio-economic and hydrologic data usually available for reservoir planning and design, the replacement of short memory models with long memory ones in reservoir analyses, cannot be objectively justified.

A statistical approach for discriminating between short and long memory models is to use the AIC of Section 6.3. The AIC provides a means of model discrimination based on the principles of good statistical fit and parsimony of the model parameters. For the six annual riverflow time series considered in Section 10.4.7 the results of Table 10.4.4 show that in all six cases the AIC chooses the best fitting ARMA model in preference to the FGN process. Although there may be certain situations where the FGN model is appropriate to use, the inherent inflexibility of a FGN process may limit the use of this model in many types of practical applications. Rather than allowing for a choice of the required number of model parameters to use in a given situation as is done in ARMA modelling, the FGN model is always restricted to just three parameters (i.e., the mean, the variance, and H).

By adhering to the model construction stages of Part III, it is a straightforward procedure to develop an appropriate ARMA to describe a particular time series. If the phenomenon being modelled has been influenced significantly by external interventions, these effects can be incorporated into the model using the intervention model of Chapter 19. By employing Monte Carlo techniques, the ECDF's of statistics such as the RAR or K can be developed to any desired accuracy, as shown in Section 10.6.3. The ECDF's are used in conjunction with a specified statistical test to check for the preservation of historical statistics in Section 10.6.4. This testing procedure can be used to check for the retention of any observed statistics by ARMA or by other types of stochastic models. Tsay (1992), for example, employs a similar approach for investigating the reproducibility of historical statistics by fitted models.

Besides considering Hurst's estimate K of the coefficient h, it is possible to entertain other types of estimates as explained in Section 10.7. For the 23 natural time series listed in Tables 10.6.1 and 10.6.2, the Siddiqui coefficient SH (Siddiqui, 1976) and Gomide's statistic YH (Gomide, 1975) possess a mean value less than K. By examining the standard deviations of the YH and SH statistics the Hurst phenomenon is seen to be less pronounced for these estimates than it is for K.

If one wishes to consider fitting a long memory model to a specified time series, one may wish to entertain the fractional ARMA or FARMA model as an alternative to FGN. This model is more flexible to use than the long memory FGN model because the number of model parameters is not fixed. In fact, as shown in the next chapter, the FARMA model is a direct extension of ARMA and ARIMA models.

APPENDIX A10.1

REPRESENTATIVE

EMPIRICAL CUMULATIVE DISTRIBUTION FUNCTIONS

(ECDF's)

FOR HURST STATISTICS

The three tables presented in this appendix contain ECDF's for simulation studies explained in Section 10.6.3. More specifically, for a range of lengths N of simulated sequences the following three sets of ECDF's are given:

Table A10.1.1. ECDF's of K for a NID$(0,\sigma_a^2)$ process.

Table A10.1.2. ECDF's of K for a Markov process with $\phi_1 = 0.4$.

Table A10.1.3. ECDF's of $\bar{R}^*_N$ for a Markov process $\phi_1 = 0.4$.

Table A10.1.1. ECDF's of K for a NID$(0,\sigma_a^2)$ process.*

Value of N	0.005	0.010	0.025	0.050	0.100	0.200	0.300	0.400	...
5	0.260	0.294	0.354	0.41?	0.485	0.474	0.634	0.677	
10	0.355	0.380	0.422	0.450	0.506	0.566	0.611	0.648	
15	0.390	0.414	0.445	0.478	0.518	0.570	0.605	0.637	
20	0.406	0.426	0.459	0.486	0.520	0.565	0.600	0.628	
25	0.414	0.436	0.466	0.497	0.527	0.569	0.509	0.626	
30	0.429	0.445	0.473	0.501	0.531	0.570	0.598	0.622	
35	0.436	0.453	0.482	0.505	0.534	0.569	0.596	0.619	
40	0.446	0.463	0.488	0.511	0.537	0.570	0.596	0.617	
45	0.447	0.464	0.487	0.519	0.537	0.570	0.594	0.615	
50	0.453	0.468	0.489	0.519	0.537	0.570	0.593	0.613	
60	0.461	0.473	0.494	0.515	0.538	0.569	0.591	0.610	
70	0.460	0.474	0.498	0.515	0.538	0.568	0.589	0.607	
80	0.461	0.475	0.499	0.518	0.450	0.568	0.588	0.606	
90	0.464	0.483	0.501	0.519	0.540	0.569	0.588	0.604	
100	0.466	0.480	0.499	0.520	0.541	0.566	0.585	0.601	
125	0.474	0.489	0.507	0.523	0.542	0.565	0.583	0.598	
150	0.477	0.489	0.506	0.523	0.542	0.564	0.581	0.596	
175	0.476	0.488	0.508	0.523	0.540	0.563	0.579	0.594	
200	0.484	0.494	0.509	0.525	0.542	0.564	0.479	0.593	
500	0.491	0.497	0.512	0.524	0.538	0.556	0.559	0.580	
1000	0.494	0.502	0.514	0.525	0.537	0.552	0.594	0.573	

*Table continues on opposite page.

(Table A10.1.1 continued.)

... Quantile 0.500	0.600	0.700	0.800	0.900	0.950	0.975	0.990	0.995
0.714	0.748	0.801	0.854	0.904	0.932	0.948	0.961	0.967
0.685	0.720	0.755	0.791	0.836	0.872	0.895	0.917	0.932
0.668	0.696	0.724	0.757	0.800	0.832	0.855	0.882	0.897
0.655	0.681	0.708	0.738	0.775	0.804	0.827	0.852	0.870
0.651	0.674	0.698	0.726	0.762	0.790	0.813	0.836	0.849
0.644	0.666	0.689	0.716	0.750	0.777	0.799	0.822	0.835
0.640	0.661	0.683	0.709	0.743	0.768	0.790	0.809	0.823
0.637	0.656	0.679	0.702	0.734	0.759	0.780	0.801	0.815
0.634	0.654	0.675	0.699	0.730	0.753	0.774	0.791	0.806
0.632	0.650	0.669	0.692	0.722	0.748	0.766	0.786	0.801
0.627	0.645	0.664	0.684	0.714	0.735	0.751	0.772	0.786
0.623	0.641	0.658	0.678	0.706	0.728	0.746	0.765	0.777
0.622	0.638	0.655	0.675	0.702	0.723	0.740	0.757	0.769
0.619	0.635	0.652	0.671	0.696	0.716	0.723	0.754	0.758
0.616	0.632	0.648	0.666	0.691	0.711	0.728	0.746	0.761
0.613	0.627	0.642	0.660	0.684	0.703	0.718	0.732	0.744
0.609	0.624	0.638	0.555	0.678	0.697	0.713	0.730	0.740
0.607	0.620	0.634	0.650	0.672	0.689	0.704	0.720	0.729
0.606	0.619	0.632	0.647	0.668	0.685	0.698	0.713	0.721
0.591	0.602	0.613	0.626	0.643	0.657	0.669	0.683	0.693
0.583	0.592	0.602	0.614	0.630	0.642	0.653	0.664	0.671

...

Table A10.1.2. ECDF's of K for a Markov process with $\phi_1 = 0.4$.*

Value of N	0.005	0.010	0.025	0.050	0.100	0.200	0.300	0.400	...
5	0.236	0.325	0.407	0.476	0.551	0.635	0.636	0.724	
10	0.401	0.442	0.494	0.537	0.590	0.660	0.703	0.746	
15	0.470	0.494	0.536	0.503	0.612	0.666	0.705	0.738	
20	0.435	0.507	0.545	0.580	0.618	0.665	0.700	0.730	
25	0.495	0.512	0.555	0.586	0.622	0.667	0.700	0.726	
30	0.514	0.530	0.566	0.594	0.628	0.667	0.697	0.721	
35	0.523	0.544	0.573	0.593	0.627	0.664	0.693	0.716	
40	0.530	0.550	0.577	0.603	0.631	0.666	0.591	0.714	
45	0.536	0.551	0.578	0.603	0.631	0.664	0.690	0.711	
50	0.540	0.555	0.532	0.692	0.629	0.662	0.688	0.707	
60	0.542	0.559	0.582	0.602	0.628	0.661	0.683	0.703	
70	0.546	0.553	0.583	0.603	0.626	0.658	0.679	0.698	
80	0.542	0.559	0.583	0.605	0.628	0.655	0.677	0.695	
90	0.549	0.562	0.585	0.604	0.626	0.656	0.676	0.692	
100	0.550	0.563	0.585	0.605	0.625	0.651	0.671	0.688	
125	0.558	0.568	0.588	0.605	0.624	0.649	0.667	0.683	
150	0.555	0.564	0.534	0.602	0.622	0.646	0.663	0.673	
175	0.552	0.567	0.586	0.601	0.619	0.642	0.660	0.674	
200	0.559	0.570	0.586	0.602	0.620	0.642	0.658	0.672	...

*Table continues on opposite page.

(Table A10.1.2 continued.)

Quantile 0.500	0.600	0.700	0.800	0.900	0.950	0.975	0.990	0.995
0.763	0.810	0.851	0.835	0.922	0.943	0.958	0.967	0.971
0.777	0.806	0.833	0.863	0.894	0.916	0.932	0.948	0.958
0.766	0.792	0.819	0.845	0.877	0.901	0.918	0.931	0.941
0.756	0.778	0.804	0.830	0.862	0.884	0.900	0.916	0.927
0.750	0.773	0.795	0.820	0.851	0.872	0.889	0.906	0.914
0.744	0.765	0.787	0.811	0.841	0.863	0.880	0.898	0.905
0.737	0.759	0.780	0.803	0.833	0.854	0.872	0.887	0.898
0.734	0.754	0.774	0.796	0.826	0.847	0.864	0.830	0.893
0.731	0.749	0.770	0.793	0.821	0.841	0.857	0.873	0.885
0.726	0.744	0.763	0.785	0.813	0.836	0.852	0.870	0.879
0.720	0.738	0.756	0.776	0.803	0.823	0.838	0.855	0.866
0.715	0.731	0.749	0.769	0.795	0.815	0.831	0.849	0.859
0.712	0.728	0.745	0.765	0.790	0.809	0.826	0.841	0.851
0.703	0.724	0.740	0.759	0.784	0.803	0.818	0.836	0.847
0.704	0.719	0.735	0.753	0.778	0.797	0.813	0.829	0.840
0.697	0.712	0.727	0.745	0.763	0.786	0.801	0.816	0.825
0.692	0.706	0.721	0.738	0.761	0.780	0.795	0.810	0.813
0.688	0.701	0.715	0.732	0.752	0.771	0.785	0.799	0.809
0.685	0.698	0.712	0.727	0.748	0.764	0.777	0.792	0.801

Table A10.1.3. ECDF's of $\bar{R}^*_n$ for a Markov process with $\phi_1 = 0.4$.*

Value of N	0.005	0.010	0.025	0.050	0.100	0.200	0.300	0.400	...
5	1.30	1.35	1.45	1.55	1.66	1.79	1.88	1.94	
10	1.91	2.04	2.21	2.37	2.59	2.89	3.13	3.32	
15	2.58	2.71	2.95	3.14	3.43	3.82	4.14	4.42	
20	3.05	3.22	3.51	3.80	4.15	4.62	5.01	5.37	
25	3.49	3.73	4.06	4.39	4.82	5.39	5.86	6.26	
30	4.02	4.20	4.63	4.99	5.48	6.03	6.60	7.05	
35	4.47	4.75	5.15	5.54	6.01	6.69	7.27	7.76	
40	4.89	5.19	5.63	6.09	6.63	7.35	7.93	8.50	
45	5.30	5.55	6.06	6.53	7.13	7.91	8.57	9.15	
50	5.68	5.97	6.51	6.94	7.58	8.41	9.15	9.75	
60	6.33	6.70	7.23	7.75	8.47	9.46	10.21	10.92	
70	6.96	7.26	7.95	8.54	9.27	10.38	11.17	11.94	
80	7.38	7.85	8.60	9.30	10.13	11.20	12.16	12.98	
90	8.08	8.49	9.26	9.98	10.86	12.13	13.00	13.91	
100	8.59	9.06	9.85	10.68	11.52	12.78	13.83	14.74	
125	10.06	10.48	11.38	12.21	13.23	14.65	15.74	16.85	
150	10.99	11.43	12.47	13.47	14.65	16.28	17.52	18.71	
175	11.81	12.60	13.72	14.66	15.91	17.65	19.09	20.39	
200	13.15	13.77	14.84	15.99	17.39	19.21	20.71	22.10	

*Table continues on opposite page.

(Table A10.1.3 continued.)

Quantile 0.500	0.600	0.700	0.800	0.900	0.950	0.975	0.990	0.995
2.01	2.10	2.18	2.25	2.33	2.37	2.40	2.43	2.43
3.49	3.66	3.82	4.01	4.22	4.37	4.48	4.60	4.67
4.68	4.94	5.20	5.49	5.85	6.14	6.36	6.53	6.65
5.70	6.00	6.37	6.77	7.28	7.65	7.95	8.24	8.45
6.64	7.04	7.46	7.92	8.57	9.05	9.45	9.35	10.05
7.49	7.95	8.44	8.98	9.75	10.34	10.85	11.38	11.64
8.25	8.79	9.32	9.96	10.84	11.54	12.14	12.68	13.08
9.02	9.56	10.15	10.86	11.87	12.65	13.29	13.98	14.53
9.72	10.31	19.99	11.80	12.87	13.73	14.43	15.17	15.72
10.34	10.95	11.65	12.52	13.71	14.73	15.54	16.45	16.95
11.57	12.30	13.07	14.03	15.36	16.43	17.31	18.32	19.04
12.70	13.47	14.32	15.41	16.90	18.16	19.22	20.44	21.23
13.82	14.65	15.62	16.80	18.44	19.81	21.02	22.28	23.05
14.80	15.71	16.72	17.97	19.76	21.27	22.54	24.09	25.18
15.68	16.65	17.76	19.01	20.97	22.64	24.02	25.64	26.72
17.88	18.97	20.21	21.79	23.97	25.77	27.42	29.26	30.27
19.86	21.12	22.53	24.24	26.72	28.98	30.92	33.05	34.25
21.71	23.00	24.50	26.42	28.91	31.44	33.51	35.62	37.22
23.42	24.89	26.54	28.46	31.33	33.69	35.81	38.40	39.95

PROBLEMS

10.1 Read Hurst's (1951, 1956) original papers about his work in long term storage. Summarize what he did and comment upon his abilities as an engineer and a statistician.

10.2 In Sections 10.2 and 10.3.1, statistics are defined for studying long term storage problems. Suggest some statistics for examining short term storage problems in reservoir design.

10.3 What do you think is the most reasonable explanation for the Hurst phenomenon? Base your answer upon references given in this chapter and elsewhere.

10.4 Using equations, explain the basic mathematical design and main purposes of the shifting level models referred to in Section 10.3.3.

10.5 Mention three types of yearly time series which could be appropriately modelled by FGN models. Provide both physical and statistical justifications for your suggestions.

10.6 In Section 10.4.5, a procedure is given for claculating a one step ahead MMSE (minimum mean square error) forecast for a FGN model. Develop a formula for determining l step ahead MMSE forecasts for a FGN model where $l \geq 1$.

10.7 In Section 10.4.6, seven methods are presented for approximately simulating FGN. Select any two of these techniques and explain using equations why these methods do not exactly simulate FGN.

10.8 In Table 10.4.4, the AIC is employed to decide upon whether or not FGN or ARMA models should be used for modelling six annual riverflow time series. Carry out a similar type of study for six annual time series that are not average yearly riverflows. Comment upon the results.

10.9 Within your field of study, select a statistic which is of direct interest to you. For example, you may be a hydrologist who is interested in floods or droughts. Explain how you would carry out simulation experiments to determine whether or not time series models fitted to your data sets preserve the historical statistics that are important to you.

10.10 Carry out the simulation study that you designed in the previous question.

10.11 Summarize Tsay's (1992) approach for ascertaining whether a fitted model preserves important historical statistics. Compare Tsay's procedure to the one presented in Section 10.6.

10.12 Explain how the research of Klemes and Klemes (1988) sheds light on the Hurst phenomenon.

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