Fitting MA(q) Models in the Closed Invertible Region

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Abstract

The use of reparameterization in the maximization of the likelihood function of the MA (q) model is discussed. A general method for testing for the presence of a parameter estimate on the boundary of an MA(q) model is presented. This test is illustrated with a brief simulation experiment for the MA(q) for $q = 1, 2, 3, 4$ in which it is shown that the probability of an estimate being on the boundary increases with q.

Key Words: Admissible region for the autoregressive-moving average time series; ARMA model reparameterization; Numerical maximum likelihood estimation.

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1. Introduction

The MA (q) model with mean zero may be written, $z_t = \theta(B)a_t$, where $\theta(B) = 1 - \theta_1 B - \ldots \theta_q B^q$ and a_t is Gaussian white noise with variance σ_a^2. The model is said to be invertible if all roots of $\theta(B) = 0$ lie outside the unit circle (Box et al., 1991, §3.3.1). MA (q) models, with roots inside the unit circle, may be reparameterized so that the roots are outside the unit circle (Brockwell and Davis, 1991, p.88). When using an exact maximum likelihood algorithm, such as the innovations algorithm (Brockwell and Davis, 1991, §8.6), some parameter estimates may lie on the non-invertible boundary (Kang, 1975). Cryer and Ledolter (1981) proved algebraically that there is a positive probability of the maximum likelihood estimator being on the unit boundary in the MA (1) case. The use of reparameterization to obtain maximum likelihood estimates over the closed invertible region is discussed in §2.

A noninvertible MA (q) model means that a_t cannot be expressed in terms of past observations. For this reason, in most situations many time series analysts would prefer an invertible model. Additionally, with a noninvertible model care must be taken in the algorithms used to compute the residuals and forecasts. Statistical inference for the parameters in a noninvertible model also becomes more difficult. For these reasons it is recommended that a model be tested to determine if it has a parameter estimate on the noninvertible boundary. In those cases where a parameter estimate on the boundary is found, there are a number of remedies. One simple approach would be to consider a different type of time series model. For example, in §3 it is noted that in the mixed ARMA (p, q) case, where $p > 0$, maximum likelihood estimates on the noninvertible boundary are less likely. Another approach would be to use mean likelihood estimation (McLeod and Quenneville, 2000) or a Bayesian approach (Marriott and Smith, 1992). In addition to guaranteeing invertibility, these estimation techniques have the same first-order asymptotic efficiency as maximum likelihood and the mean-square error of the parameters estimates is usually less (McLeod and Quenneville, 2000). In §4 we present a convenient test for the presence of an estimate on the noninvertible boundary. It should be pointed out that one cannot simply constrain the maximum likelihood estimates to be inside the invertible region since from the results of

Cryer and Ledolter (1981), estimates on the boundary will cause the constrained likelihood approach to either fail by converging to an estimate outside the admissible region or else, if a penalty function approach is used for the maximization, the estimates will be very close to the boundary. Neither of these situations is desirable.

2. Reparameterization For Constrained Maximum Likelihood Estimation

Let $\mathcal{D}_\theta$ denote the invertible region for an MA (q) in the parameter space $(\theta_1, \ldots, \theta_q)$. As noted by Monahan (1984), the reparameterization discussed by Barndorff-Neilsen and Schou (1973) may be extended for use with the ARMA (p, q). For simplicity we discuss the MA (q) case but it is easy to extend our results to the moving-average parameters in the ARMA (p, q). Monahan (1984) defined a transformation, $\mathcal{B} : (\zeta_1, \ldots, \zeta_q) \longrightarrow (\theta_1, \ldots, \theta_q)$. For brevity, let $\zeta = (\zeta_1, \ldots, \zeta_q)$, $\theta = (\theta_1, \ldots, \theta_q)$ and $\theta_i = \mathcal{B}_i(\zeta), i = 1, \ldots, q$. This transformation may be computed using the recursion,

$$\theta_{i,k} = \theta_{i,k-1} + \zeta_k \theta_{k-i,k-1}, \quad i = 1, ..., k-1; k = 1, \ldots, q, \tag{1}$$

where $\theta_{i,q} = \theta_i, \zeta_i = \theta_{i,i}, i = 1, \ldots, q$. The invertible region for the transformed parameters ζ, denoted by $\mathcal{D}_\zeta$, is simply the interior of the unit cube, $|\zeta_i| < 1, i = 1, \ldots, q$. Barndorff-Neilsen and Schou (1973) showed that $\mathcal{B}$ is 1:1 and onto as well as continuously differentiable with a continuously differentiable inverse inside the invertible region. However, since the determinant Jacobian of the transformation $\mathcal{B}$ is zero on the non-invertible boundary (Barndorff-Neilsen and Schou, 1973, p.414), the transformation is not 1:1 there and consequently the inverse function is not well defined. For example in the MA (2) case, $\mathcal{B}(\zeta_1, \zeta_2) = (\zeta_1(1-\zeta_2), \zeta_2)$ and so $\mathcal{B}(\zeta_1, 1) = (0, 1)$.

Denote the likelihood function of the MA (q) by $\mathcal{L}(\theta)$. Then the reparameterized likelihood function may be written $\mathcal{L}(\mathcal{B}(\zeta))$, where ζ is constrained, $|\zeta_i| \leq 1, i = 1, \ldots, q$. Standard minimization algorithms, such as those implemented in *Mathematica* (Wolfram, 2005), may be used to obtain the maximum likelihood estimates over the closed invertible region.

The transformation $\mathcal{B}$ is continuous and differentiable in the closed invertible region. In order to use the transformation $\mathcal{B}$ for maximum likelihood estimation in the case where the estimates may lie on the boundary of $\mathcal{D}_\theta$, it is necessary that each point on the boundary of $\mathcal{D}_\theta$ be the image of a point on the boundary of $\mathcal{D}_\zeta$. In other words, it is necessary that $\mathcal{B}$ be onto. This is proved in §3.

3. Proof $\mathcal{B}$ Is Onto

Denote the boundary sets of $\mathcal{D}_\theta$ and $\mathcal{D}_\zeta$ by ∂_θ and ∂_ζ respectively. Theorem 2 of Barndorff-Nielsen and Schou (1973) showed that $\mathcal{B}$ maps $\mathcal{D}_\zeta$ onto $\mathcal{D}_\theta$ and that this mapping is one-to-one. But $\mathcal{B}$ is no longer one-to-one on the boundary ∂_ζ. It is non-trivial to show that $\mathcal{B}$ maps ∂_ζ onto ∂_θ. After careful investigation and discussion with an expert in point set topology we were not able to find any suitable theorem which is applicable to this situation and so for completeness we have included a proof from first principles.

Theorem 1. $\partial_\theta = \mathcal{B}(\partial_\zeta)$

Proof. First we show that $\partial_\theta \subset \mathcal{B}(\partial_\zeta)$. Let $\bar{\mathcal{D}}_\theta$ and $\bar{\mathcal{D}}_\zeta$ denote the closures of $\mathcal{D}_\theta$ and $\mathcal{D}_\zeta$ respectively. Since $\mathcal{B}$ is a polynomial and hence continuous on $\bar{\mathcal{D}}_\zeta$, $\mathcal{B}(\bar{\mathcal{D}}_\zeta) \subset \overline{\mathcal{B}(\mathcal{D}_\zeta)} = \bar{\mathcal{D}}_\theta$. Meanwhile, $\mathcal{B}(\bar{\mathcal{D}}_\zeta) = \mathcal{D}_\theta \cup \mathcal{B}(\partial_\zeta)$ is closed since $\bar{\mathcal{D}}_\zeta$ is compact and therefore $\mathcal{B}(\bar{\mathcal{D}}_\zeta) \supset \bar{\mathcal{D}}_\theta$. Hence $\mathcal{B}(\bar{\mathcal{D}}_\zeta) = \bar{\mathcal{D}}_\theta$, so $\mathcal{D}_\theta \cup \mathcal{B}(\partial_\zeta) = \mathcal{D}_\theta \cup \partial_\theta$. Since $\mathcal{B}$ is a homeomorphism between $\mathcal{D}_\theta$ and $\mathcal{D}_\zeta$, it follows that $\mathcal{D}_\theta$ is open since $\mathcal{D}_\zeta$ is open. $\mathcal{D}_\theta \cap \partial_\theta = \emptyset$. Hence $\partial_\theta \subset \mathcal{B}(\partial_\zeta)$.

Next, we show $\partial_\theta \supset \mathcal{B}(\partial_\zeta)$. Let $\vartheta \in \partial_\zeta$. There exists a sequence $\{\vartheta_n\} \in \mathcal{D}_\zeta$ such that $\vartheta_n \to \vartheta$. Hence $\mathcal{B}(\vartheta_n) = \vartheta_n \in \mathcal{D}_\theta \to \mathcal{B}(\vartheta) = \vartheta$ by the continuity of $\mathcal{B}$ on $\bar{\mathcal{D}}_\zeta$. If $\vartheta \in \mathcal{D}_\theta$, $\mathcal{B}^{-1}(\vartheta_n) = \vartheta_n \to \mathcal{B}^{-1}(\vartheta) = \vartheta \in \mathcal{D}_\zeta$ by the continuity of $\mathcal{B}^{-1}$ on $\mathcal{D}_\theta$, which causes a contradiction with $\vartheta \in \partial_\zeta$. Therefore, $\mathcal{B}(\vartheta) \notin \mathcal{D}_\theta$. It follows that $\mathcal{B}(\vartheta) \in \partial_\theta$ since $\mathcal{D}_\theta \cup \mathcal{B}(\partial_\zeta) = \mathcal{D}_\theta \cup \partial_\theta$. Hence $\partial_\theta \supset \mathcal{B}(\partial_\zeta)$.

4. Test for Estimate on the Noninvertible Boundary

Monahan (1984) showed that inside the invertible region, $\mathcal{B}^{-1}$, may be obtained using the recursive formula,

$$\theta_{i,k-1} = (\theta_{i,k} - \theta_{k,k}\theta_{k-i,k})/(1-\theta_{k,k}^2), \quad i = 1, \ldots, k-1; k = q, \ldots, 1, \tag{2}$$

where $\theta_i = \theta_{i,q}, \zeta_i = \theta_{i,i}, i = 1, \ldots, q$. More generally, for the closed invertible region, define $\mathcal{B}^- : \bar{\mathcal{D}}_\theta \longrightarrow \bar{\mathcal{D}}_\zeta$ using the recursion,

$$\begin{aligned}\theta_{i,k-1} &= (\theta_{i,k} - \theta_{k,k}\theta_{k-i,k})/(1-\theta_{k,k}^2), \quad \text{if } |\theta_{j,j}| < 1, \text{ for all } j \geq k, \\ &= 0 \quad \text{otherwise,}\end{aligned} \tag{3}$$

where $i = 1, \ldots, k-1; k = q, \ldots, 1$. In $\mathcal{D}_\theta$, $\mathcal{B}^-$ is the same as $\mathcal{B}^{-1}$ and, using mathematical induction, it may be shown that all points on the boundary of $\mathcal{D}_\theta$ are mapped into boundary points in $\mathcal{D}_\zeta$. As an illustration,

$$\mathcal{B}(\zeta_1, 1, \zeta_3, \zeta_4) = (-\zeta_3(\zeta_4+1), 1-\zeta_4, \zeta_3(\zeta_4+1), \zeta_4)$$

and

$$\mathcal{B}^-(-\zeta_3(\zeta_4+1), 1-\zeta_4, \zeta_3(\zeta_4+1), \zeta_4) = (0, 1, \zeta_3, \zeta_4).$$

Let $\hat{\theta}$ denote estimates which belong to the closed invertible region and let $\hat{\zeta} = \mathcal{B}^-(\hat{\theta})$, where $\hat{\theta} \in \bar{\mathcal{D}}_\theta$. Then $|\hat{\zeta}_i| \leq 1$ and $\hat{\theta}$ is on the boundary if and only if $\hat{\zeta}_i = \pm 1$ for at least one $i \in \{1, \ldots, q\}$. In practice, we need to take into account the finite precision and rounding error in our computations so we may declare an estimate is on the boundary if $|1 - |\hat{\zeta}_i|| < \epsilon$ for any $i \in \{1, \ldots, q\}$. In the simulation example below, we took $\epsilon = 10^{-6}$. If a different software environment were used it might be advisable to take ϵ somewhat larger. The use of this test is illustrated in §5.

5. Simulation Experiment

A simulation experiment was conducted to see how the probability of an estimate on the boundary depends on the model order q and series length n. An MA (1) with parameter $\theta_1 = -0.9, -0.6, \ldots, 0.9$, was simulated 100 times and for each simulation the exact maximum likelihood estimates were determined using the innovation algorithm and the `NMinimize` function in *Mathematica* (Wolfram, 2005) for each of the MA(q) models with $q = 1, 2, 3, 4$. An estimate was counted as on the boundary if any one of $\hat{\zeta}_1, \ldots, \hat{\zeta}_q$ was within 10^{-6} of ± 1. The results are summarized in Table 1. We see that the probability of an estimate on the boundary increases with the model order q, decreases with sample size n and increases with the distance of the true parameter to the unit boundary. Note that the test for an estimate on the boundary requires Theorem 1. For the model MA(1), the results agree well with the previous studies mentioned above.

In another simulation experiment, ARMA$(1, q), q = 1, 2, 3, 4$ models were fit to a series of length 25 generated by an ARMA$(1, 1)$ model with parameters $\phi = -0.5, 0, 0.5$ and $\theta_1 = -0.9, -0.6, \ldots, 0.9$. It was found that the probability of a root on the moving average boundary was quite small but appeared to be nonzero in all cases.

6. Concluding Remarks

Theorem 1 is also useful in the maximum likelihood estimation of ARMA(p, q) models, $\phi(B)z_t = \theta(B)a_t$ where $\phi(B) = 1 - \phi_1 B - \ldots \phi_p B^p$ and $\theta(B) = 1 - \theta_1 B - \ldots \theta_q B^q$. For a causal-stationary process, $\phi(B) = 0$ has all roots outside the unit circle and for model identifiability, in the econometric sense (Harvey, 1990, §3.6), we require that $\theta(B) = 0$ has all roots on or inside the unit circle. The results §2 and §4 may be extended to the case. Complete details of the maximum likelihood algorithm for ARMA(p, q) models, as well as of the simulations in §5, are given in *Mathematica* notebooks available from the authors.

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Table 1: Proportion of times an estimate was on the noninvertible boundary in 100 simulations of a series of length n from an MA (1) with parameter θ_1 when it is fit with an MA (q), $q = 1, 2, 3, 4$.

n	θ	MA (1)	MA (2)	MA (3)	MA (4)
25	-0.9	0.53	0.51	0.51	0.46
25	-0.6	0.08	0.15	0.23	0.53
25	-0.3	0.00	0.03	0.17	0.27
25	0.0	0.01	0.05	0.24	0.35
25	0.3	0.02	0.06	0.23	0.27
25	0.6	0.14	0.21	0.37	0.39
25	0.9	0.49	0.51	0.49	0.57
50	-0.9	0.36	0.36	0.37	0.32
50	-0.6	0.01	0.03	0.06	0.11
50	-0.3	0.00	0.03	0.04	0.08
50	0.0	0.00	0.00	0.02	0.04
50	0.3	0.01	0.02	0.02	0.04
50	0.6	0.01	0.03	0.03	0.07
50	0.9	0.26	0.33	0.28	0.36