Preservation of the Rescaled Adjusted Range
1. A Reassessment of the Hurst Phenomenon

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Previous research related to the controversial Hurst phenomenon is reviewed and evaluated. Because of the inherent statistical properties of the rescaled adjusted range (RAR) statistic it is suggested that research primarily be devoted to this statistic rather than to the various definitions of the Hurst coefficient. Simulation studies reveal that for independently distributed random variables the RAR does not significantly depend on the underlying distribution of the random variables but is a function of the sample size. For modeling correlated data the statistical attributes of a discrete fractional Gaussian noise (FGN) process are studied and also improved. An efficient maximum likelihood estimation technique is developed for the FGN model, and it is shown how the residuals of the fitted model can be calculated and then subjected to diagnostic checks. An exact simulation procedure is developed for simulating FGN in such a way that synthetic traces from the model lie outside the Brownian domain. The Akaike information criterion (AIC) is suggested as a method for choosing between a FGN and a Box-Jenkins model. For the six annual river flow series that are considered the AIC selects the best Box-Jenkins model in preference to the FGN process for each data set. Because Box-Jenkins models can be shown to preserve the historical RAR, in many practical applications it may be advantageous to use a Box-Jenkins model instead of a FGN process.

INTRODUCTION

Since the original empirical studies of *Hurst* [1951] the Hurst phenomenon has caused extensive research with accompanying academic controversies right up to the present time. The purpose of this paper is to review, to appraise, and to suggest how to improve the research related to Hurst's work. The views presented in this paper and the two associated articles [*Hipel and McLeod*, 1978*a*, *b*] in some ways represent a fresh approach to the study of the Hurst phenomenon and the related problem of the preservation of historical statistics by stochastic models.

Definitions related to the cumulated range first are clearly defined. Then the various types of Hurst coefficients that have been developed for use in formulae involving the rescaled readjusted range (RAR) are evaluated. Because of the flexible statistical properties of the RAR it is suggested that this is the Hurst statistic of primary concern in water resource applications related to storage.

The roles of both identically independently distributed (IID) variables and correlated random variables for explaining problems related to the Hurst phenomenon are thoroughly investigated. Simulation studies are used to demonstrate that the RAR is nearly independent of the type of underlying distribution of the random variables and is also a function of the sample size. Of particular importance for correlated processes are stochastic models that can easily be fit to geophysical time series and at the same time retain relevant historical statistical characteristics of the data such as the RAR and other related statistics. Box-Jenkins models [*Box and Jenkins*, 1970] constitute one family of stochastic models which possess the potential for continued extensive utilization in hydrology. The discrete fractional Gaussian noise (FGN) model is a process that was developed mainly within the hydrological literature [*Mandelbrot and Wallis*, 1968, 1969*a*, *b*, *c*, *d*, *e*] as a means for possibly accounting for the Hurst phenomenon. Although some of the inherent drawbacks of this model are discussed, significant contributions are formulated toward the further statistical maturity of the FGN model.

When any type of stochastic model is being developed to model a given time series it is recommended to follow the identification, estimation, and diagnostic check stages of model construction [*Box and Jenkins*, 1970; *Box and Tiao*, 1973]. Because this modeling philosophy previously has not been adhered to strictly for a FGN process, important developments are presented for this model. An efficient maximum likelihood (ML) procedure is derived for use at the estimation stage. Simulation studies reveal that the ML approach is superior to a previous estimation method. A technique for calculating the model residuals is presented, so that the statistical properties of the residuals can be tested by specified diagnostic checks. If, for example, the residuals fail to pass the whiteness criterion, another type of model should be chosen in order to satisfy this important modeling assumption. Finally, an exact simulation procedure is given for simulating FGN, and the use of this technique is demonstrated in a simulation study. This new method eliminates the need for approximating FGN by other types of stochastic processes. The standard Fortran computer algorithms and accompanying documentation for these improvements in FGN modeling are listed by *Hipel and McLeod* [1978*b*] (part 3 of this set).

In Box-Jenkins modeling, additional procedures have recently been presented for use at the three modeling stages [*Hipel et al.*, 1977*a*], and these advanced techniques have successfully been applied to water resource problems [*McLeod et al.*, 1977]. If one has to select between a Box-Jenkins or a FGN process to model a given data set, the Akaike information criterion (AIC) [*Akaike*, 1974] is suggested as a means of model discrimination. For the six annual river flow time series considered, the AIC selects the Box-Jenkins model in preference to the FGN process in each case. Fitting Box-Jenkins

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autoregressive moving average (Arma) models to 23 geophysical time series and using Monte Carlo techniques, *Hipel and McLeod* [1978a] demonstrate in part 2 of this set that Box-Jenkins models do preserve the RAR. Because of the aforementioned facts, in many practical situations it may be advisable to utilize Arma models rather than FGN processes.

Definitions

Consider a time series $z_1, z_2, \cdots, z_N$. Define the kth general partial sum as

$$S_k' = S_{k-1}' + (z_k - \alpha\bar{z}_N) = \sum_{i=1}^{k} z_i - \alpha k\bar{z}_N \qquad (1)$$

$$k = 1, 2, \cdots, N$$

where S_0' equals 0, $\bar{z}_N$ equals $1/N\sum_{i=1}^{N} z_i$, the mean of the first N terms of the series, and α is a constant satisfying $0 \leq \alpha \leq 1$. The general (cumulative) range R_N' is defined as

$$R_N' = M_N' - m_N' \qquad (2)$$

where M_N' equals max $(0, S_1', S_2', \cdots, S_N')$, the general surplus, and m_N' equals min $(0, S_1', S_2', \cdots, S_N')$, the general deficit. Thus R_N' is the range of cumulative departures of the random variables $z_1, z_2, \cdots, z_N$ from $\alpha\bar{z}_N$. When random variables such as $z_1, z_2, \cdots, z_N$ are employed in summation operations, they are often referred to as summands. The rescaled general range $\tilde{R}_N'$ is given as

$$\tilde{R}_N' = R_N'/D_N' \qquad (3)$$

where D_N' equals $N^{-1/2}[\sum_{i=1}^{N}(z_i - \alpha\bar{z}_N)^2]^{1/2}$ is the general deviation.

The constant α can be thought of as an adjustment factor, or in storage theory, it can be thought of as the degree of development of reservoir design. Two special cases for α are of particular importance in water resources. For $\alpha = 0$ (no adjustment) the kth general partial sum S_k' is replaced by the crude partial sum S_k, which is defined by

$$S_k = S_{k-1} + z_k = \sum_{i=1}^{k} z_i \qquad k = 1, 2, \cdots, N \qquad (4)$$

where S_0 equals 0. The crude range R_N is defined analogous to R_N' as

$$R_N = M_N - m_N \qquad (5)$$

where M_N equals max $(0, S_1, S_2, \cdots, S_N)$, the crude surplus, and m_N equals min $(0, S_1, S_2, \cdots, S_N)$, the crude deficit. Similarly, the rescaled crude range is

$$\tilde{R}_N = R_N/D_N \qquad (6)$$

where D_N equals $N^{-1/2}[\sum_{i=1}^{N} z_i^2]^{1/2}$ is the crude deviation.

When $\alpha = 1$ (maximum adjustment or development) the kth adjusted partial sum S_k^* is given by

$$S_k^* = S_{k-1}^* + (z_k - \bar{z}_N) = \sum_{i=1}^{k} z_i - k\bar{z}_N \qquad (7)$$

$$k = 1, 2, \cdots, N$$

where S_0^* equals 0 and S_N^* equals 0. The adjusted range R_N^* is defined as

$$R_N^* = M_N^* - m_N^* \qquad (8)$$

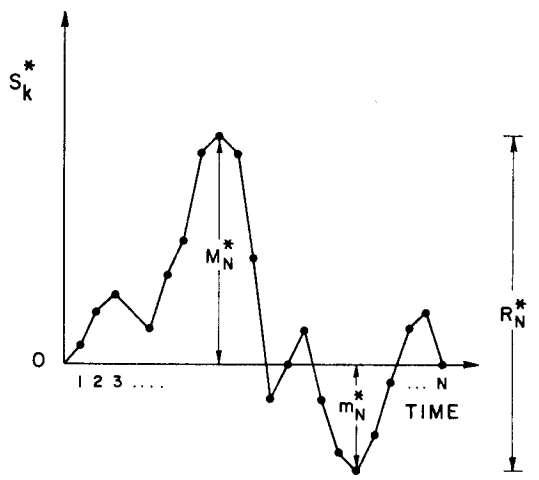

Fig. 1. Adjusted range.

where M_N^* equals max $(0, S_1^*, S_2^*, \cdots, S_N^*)$, the adjusted surplus, and m_N^* equals min $(0, S_1^*, S_2^*, \cdots, S_N^*)$, the adjusted deficit. Finally, the rescaled adjusted range is

$$\tilde{R}_N^* = R_N^*/D_N^* \qquad (9)$$

where D_N^* equals $N^{-1/2}[\sum_{i=1}^{N}(z_i - \bar{z}_N)^2]^{1/2}$ is the sample standard deviation. Figure 1 graphically illustrates the concepts of S_k^*, M_N^*, m_N^*, and R_N^*.

The statistics described in this section are extremely useful in reservoir design. If the z_t are average annual volumes of river flow, then $\sum_{i=1}^{k} z_i$ is the inflow into a reservoir in k years, and $\alpha k\bar{z}_N$ is the outflow at a level of development α. S_k' in (1) represents the storage after k years. R_N' is the minimum reservoir capacity required to satisfy a constant draft of $\alpha\bar{z}_N$ without experiencing shortages or spills over the period spanned by the inflow sequence $z_1, z_2, \cdots, z_N$. When $\alpha = 1$, the water in the river would be used to its full potential.

The time series $z_1, z_2, \cdots, z_N$ is said to be covariance stationary if the mean

$$\mu = E(z_t) \qquad (10)$$

and the theoretical autocovariance function (TACVF)

$$\gamma_k = E[(z_t - \mu)(z_{t-k} - \mu)] \qquad (11)$$

both exist and do not depend on t. The statistical properties of any covariance stationary Gaussian time series are completely determined by its mean μ, variance γ_0, and theoretical autocorrelation function (TACF),

$$\rho = \gamma_k/\gamma_0 \qquad (12)$$

Klemes [1974] discusses the physical interpretations of the stationary assumptions.

Often it seems reasonable to assume that recent values of the time series contain more information about the present and future values than about those values in the remote past. Accordingly, it is assumed that the TACVF is summable as defined by [*Brillinger*, 1975]

$$\sum_{k=-\infty}^{\infty} |\gamma_k| < \infty \qquad (13)$$

A covariance stationary time series model is said to have a short or a long memory according to whether the TACVF (or equivalently the TACF) is summable. Thus the FGN model

has a long memory (for the parameter H in the range $0.5 < H < 1$), whereas the Arma models have a short memory.

Historical Research

The Hurst Phenomenon and Hurst Coefficients

Hurst [1951, 1956] stimulated interest in the RAR statistic by his studies of long-term storage requirements on the Nile River. On the basis of a study of 690 annual time series comprising streamflow, river and lake levels, precipitation, temperature, pressure, tree ring, mud varve, sunspot, and wheat price records, Hurst implied that $\bar{R}_N^*$ varies with N as

$$\bar{R}_N^* \propto N^h \tag{14}$$

where h is the generalized Hurst coefficient. The above equation can be written in the general form

$$\bar{R}_N^* = aN^h \tag{15}$$

where a is a coefficient that is not a function of N. It should be noted that Hurst did not explicitly state the generalized Hurst law of (15) in his research papers. However, by choosing the coefficient a to have a value of $(\frac{1}{2})^h$, Hurst in effect estimated h by the Hurst coefficient K in the empirical equation

$$\bar{R}_N^* = (N/2)^K \tag{16}$$

By taking logarithms of (16), an explicit relationship for K is then

$$K = \frac{\log \bar{R}_N^*}{\log N - \log 2} = \frac{\log R_N^* - \log D_N^*}{\log N - \log 2} \tag{17}$$

Employing series that varied in length from 30 to 2000 years, Hurst found K to range from 0.46 to 0.96 with a mean of 0.73 and a standard deviation of 0.09.

Assuming a normally independently distributed (NID) process, *Hurst* [1951] utilized some coin-tossing experiments to develop the theoretical asymptotic relationship for the expected value of the adjusted range as

$$E(R_N^*) = (\pi N \gamma_0/2)^{1/2}$$

or

$$E(R_N^*)/(\gamma_0)^{1/2} = 1.2533N^{1/2} \tag{18}$$

Using the theory of Brownian motion, *Feller* [1951] rigorously established the above asymptotic formula for any sequence of IID random variables possessing finite variance. It follows from a standard convergence theorem in probability theory [*Rao*, 1973, p. 122] that for large N,

$$E(\bar{R}_N^*) = 1.2533N^{1/2} \tag{19}$$

Even though Hurst studied the RAR for small N and not for the adjusted range, the form of (18) prompted him to use K in (17) as an estimate of h and also to assume K to be constant over time. However, for 690 geophysical time series, Hurst found K to have an average of 0.73, while the asymptotic, or limiting, value of K given by (19) is 0.5. This discrepancy is referred to as the Hurst phenomenon. The search for a reasonable explanation of the Hurst phenomenon and the need for methods whereby the statistics related to Hurst's work can be incorporated into mathematical models have intrigued researchers during the past quarter of a century.

In addition to K, other estimates of the generalized Hurst coefficient h in (15) have been formulated. Based upon the structure of (19), *Gomide* [1975] suggested estimating h by the YH that is given in the following equation:

$$\bar{R}_N^* = 1.2533N^{YH} \tag{20}$$

The average value of YH for the 690 series considered by Hurst is 0.57 rather than 0.73.

Siddiqui [1976] proposed a method of evaluating h if the underlying process is assumed to be an Arma process. The estimate of Siddiqui is the result of a comparison between an asymptotic result for calculating $E(\bar{R}_N^*)$ for Arma processes and the form of (15). Siddiqui's estimate of h and the statistic YH of *Gomide* [1975] are calculated for the 23 geophysical time series considered by *Hipel and McLeod* [1978*a*]. Appropriate conclusions are drawn regarding the behavior of these statistics in relationship to K and whether they exhibit the Hurst phenomenon. For the case of a white noise process, Siddiqui's estimate of h is identical with Gomide's statistic YH in (20).

For NID random variables, *Anis and Lloyd* [1976] have suggested a specific estimate of h that is a function of the sample size. By taking logarithms of (15) for the expected value of the RAR, the following equation is obtained.

$$\log E(\bar{R}_N^*) = \log a + h \log N \tag{21}$$

Anis and Lloyd [1976] defined the local Hurst exponent $h(N)$ as the derivative

$$h(N) = \partial[\log E(\bar{R}_N^*)]/\partial(\log N) \tag{22}$$

The exponent $h(N)$ can be tabulated approximately from the equation

$$h(N) = \frac{\log E(\bar{R}_{N+1}^*) - \log E(\bar{R}_{N-1}^*)}{\log (N+1) - \log (N-1)} \tag{23}$$

where $E(\bar{R}_N^*)$ is calculated exactly by using the formula of *Anis and Lloyd* [1976] that is also given in (28). It should be noted that previously *Salas-La Cruz and Boes* [1974] had defined an exponent similar to $h(N)$ for the general range where $0 \leq \alpha \leq 1$.

Because the entries for the expected value of the RAR on the right-hand side of (23) are calculated directly from a theoretical formula, $h(N)$ is not a function of the data and is therefore not a statistic. Nevertheless, it would be possible perhaps to fit some type of stochastic model to a given time series and then to derive the RAR terms in (23) by using simulation. Most likely, this type of procedure may not be a worthwhile venture, and hence $h(N)$ probably will have limited use in practical hydrological problems.

Anis and Lloyd [1976, p. 115, Table 1] list values of $h(N)$ for N ranging from 5 to 10^6. Although the magnitude of $h(N)$ asymptotically approaches 0.5 for increasing N, at lower values of N, $h(N)$ is significantly larger than 0.5. For instance, when N possesses values of 5, 40, 100, 200, and 500, $h(N)$ has magnitudes of 0.6762, 0.5672, 0.5429, 0.5315, and 0.5202, respectively.

In the development of an estimate for the parameter H in FGN models, *Mandelbrot and Wallis* [1969*d*] assumed a form of the Hurst law that is identical with (15). For a given time series $z_1, z_2, \cdots, z_N$, let $\bar{R}_{r'}^*(t, r)$ denote the RAR of the subseries $z_t, z_{t+1}, \cdots, z_r$ and let $r' = r - t + 1$. When examining scatter plots (or 'pox diagrams') of $\log \bar{R}_{r'}^*(t, r)$ versus $\log r'$ for a number of selected values of t and r, *Mandelbrot and*

Wallis [1969*d*] were using for each subseries a Hurst law given by

$$\bar{R}_{r'}^*(t, r) = a(r')^h \tag{24}$$

Wallis and Matalas [1970] have suggested the G Hurst estimator for estimating the parameter H in FGN models and also h in (24). This procedure estimates h by calculating the slope of the regression of the averaged values of $\log \bar{R}_{r'}^*(t, r)$ on $\log r'$ for specified values of t and r.

When Hurst originally formulated (16) there is no doubt that he was attempting to derive an empirical law that would be valid for a wide range of geophysical phenomena. In particular, an equation such as (16) would be extremely useful for reservoir design if the phenomenon being modeled were average annual river flows. However, the distribution of K plus the other types of Hurst exponents summarized in this section are a function of the sample size N. For example, in part 2, *Hipel and McLeod* [1978*a*] give the empirical cumulative distribution functions of K for various values of N for certain types of Arma processes. In addition, as shown by *Hipel and McLeod* [1978*a*, Table 4], when K is estimated for 23 given geophysical time series, K seldom has exactly the same value for any given pair of data sets. Because of the aforementioned facts the empirical law of Hurst in (16) loses much of its simplicity and also its potential for being a universal law. This inherent lack of universality of Hurst's law may be due to the fact that the general form of (16) resembles the asymptotic formula given in (19), whereas in practice it is necessary to deal with small and moderate sample sizes.

Because the RAR possesses many attractive statistical features, Hurst perhaps should have concentrated his efforts on studying the properties of $\bar{R}_N^*$ rather than those of K. The RAR statistic is independent of the magnitude of the mean level and standard deviation of a time series. If the data are modeled by an Arma process, $\bar{R}_N^*$ is only a function of the sample size N and the autoregressive (AR) and moving average (MA) parameters and is independent of the variance of the innovations [*Hipel*, 1975, Appendix B]. From (17) it can be seen that K is simply a transformation of $\bar{R}_N^*$ and therefore also possesses the aforementioned properties of the RAR. Nevertheless, the formulation of K in (17) as a function of $\bar{R}_N^*$ only introduces an unnecessary transformation and does not give K any additional advantageous statistical properties that are not already possessed by the RAR. The authors of this paper therefore recommend that future research should concentrate on the RAR rather than on the various types of Hurst exponents discussed in this section.

Because the concept of the Hurst coefficient is so entrenched in the literature, it is widely quoted in the remainder of this paper. The reader should be aware that the statistic of primary concern is the RAR. Even the use of the G Hurst statistic [*Wallis and Matalas*, 1970], which was primarily developed as an estimate for the parameter H of FGN models, is questionable. It is demonstrated later in this article that a maximum likelihood estimate (MLE) of H is a more efficient procedure to employ.

The Hurst Phenomenon and Independent Summands

Besides the results of *Feller* [1951], Hurst's work influenced other researchers to develop theoretical derivations for statistics related to the cumulative range. Because of the mathematical complexity in deriving theoretical formulae for the moments of statistics connected with the range a large portion of the research was devoted to the special case of independent summands. *Anis and Lloyd* [1953] developed a formula for the expected value of the crude range for standard NID variates. *Anis* [1955] derived the variance of M_N and subsequently a method for obtaining all the moments of M_N [*Anis*, 1956].

Solari and Anis [1957] determined the mean and variance of the adjusted range for a finite number of NID summands. *Feller* [1951] had noted that the sampling properties of the adjusted range were superior to those of the crude range. The results of *Solari and Anis* [1957] for the variance of M_N^* substantiated the conclusion of Feller when this variance was compared to that of M_N [*Anis*, 1956].

Moran [1964] initiated a new line of development when he observed that the expected value of cumulative ranges could easily be derived from a combinatorial result known as Spitzer's lemma. He showed that for moderate N, distributions with very large second moments about the mean could cause the $E(M_N)$ to increase more quickly than $N^{1/2}$. This, in turn, implied that the crude range would do likewise.

For independently stably distributed summands with the characteristic exponent ν, *Boes and Salas-La Cruz* [1973] showed that asymptotically

$$E(\bar{R}_N^*) \propto N^{1/\nu} \tag{25}$$

where $1 < \nu \leq 2$. The general stable distribution with characteristic exponent ν is defined for $1 < \nu \leq 2$ in terms of its characteristic function

$$\omega(u) = E(e^{iuz_t}) \tag{26}$$

by

$$\log \omega(u) = i\mu u + \sigma^2|u|^\nu\{1 + i\beta(u/|u|) \tan [(\pi/2)\nu]\} \tag{27}$$

where

$i = (-1)^{1/2}$.

μ location parameter for the random variable z_t;
σ scale parameter for the random variable z_t;
β measure of skewness.

For $\beta = 0$ and $\nu = 2$ the normal distribution is obtained. Stable distributions with characteristic exponent $1 < \nu < 2$ generate more extreme observations than the normal distribution. *Granger and Orr* [1972] have suggested that economic time series are best modeled by a stable distribution with characteristic exponent $1.5 < \nu < 2$. From (25) it could be suggested that a stable distribution with $\nu = 1.37$ (approximately) for geophysical time series could explain Hurst's findings. However, because for the case of stable distributions with $1 < \nu < 2$ the sample variance is not a consistent estimator of the scale parameter σ, it does not follow that (25) will hold for the RAR. In fact, simulation experiments reported later in this paper show that the expected value of the RAR for independently stably distributed summands with characteristic exponent $\nu = 1.3$ very nearly equals the expected value of the RAR for NID summands.

All the aforesaid research was influenced by the original work of Hurst. However, mathematicians have for a long time been investigating the crude range of independent summands independently of Hurst's empirical research. *Anis and Lloyd* [1976] give a brief survey of mathematical studies of the crude range. Further references can also be found in a paper by *Berman* [1964].

Unfortunately, none of the foregoing theoretical investigations discussed in this section have dealt with the RAR. However, for a NID process, *Anis and Lloyd* [1976] have success-

fully proved the following exact equation to be the expected value of the RAR:

$$E(\tilde{R}_N^*) = \frac{\Gamma[\frac{1}{2}(N-1)]}{(\pi)^{1/2}\Gamma(\frac{1}{2}N)} \sum_{r=1}^{N-1} \frac{N-r}{r} \tag{28}$$

The Hurst Phenomenon and Correlated Summands

When *Hurst* [1951] theoretically derived (18) for the adjusted range, he assumed normality of the process, he developed that equation as an asymptotic relationship, and he assumed independence of the time series. As was pointed out by *Wallis and Matalas* [1970], these three facts respectively caused the following three possible explanations of the Hurst phenomenon: (1) nonnormality of the probability distribution underlying the time series, (2) transience (i.e., N is not large enough for the Hurst coefficient to attain its limiting value of 0.5), and (3) autocorrelation due to nonindependence.

For independent summands, nonnormality of the underlying process has largely been discounted as a possible explanation of the Hurst phenomenon. If a very large sample is being considered, the asymptotic expression in (19) has been shown to be valid for IID random variables. For samples of small and moderate length, simulation studies later in this paper (see Table 5) reveal that the RAR is very nearly independent of the distribution of the random variables. Because the Hurst coefficient K is definitely a function of N for independent summands (see, for example, Table 6 for the NID case), then transience constitutes a plausible explanation to Hurst's dilemma.

For the autocorrelated case, *Wallis and O'Connell* [1973] correctly concluded that transience is obviously connected with the autocorrelation structure of the generating process, and therefore these two effects must be considered simultaneously when attempting to account for the Hurst phenomenon. As is illustrated by simulation studies in part 2 [*Hipel and McLeod,* 1978*a*] for Arma models, both transience and autocorrelation form an explanation of the Hurst phenomenon. In this section the roles of both short-memory and long-memory processes for explaining and modeling the Hurst phenomenon are examined.

Hurst [1951] actually conjectured that K had a value of 0.73 and not 0.5 because of persistence. This is the tendency for high values to be followed by high values and low values by low values. Persistence is caused by the dependence of naturally occurring time series as exhibited in their serial correlation structure. For reservoir design this means that for a given value of N the size of a reservoir that releases the mean flow each year would need to be larger than the capacity corresponding to an uncorrelated series of inflows.

Short-memory models. *Barnard* [1956] and *Moran* [1959] observed that for the standard short-memory time series models the following asymptotic formula is valid:

$$E(\tilde{R}_N^*) = aN^{1/2} \tag{29}$$

where a is a coefficient that does not depend on N. *Mandelbrot and Van Ness* [1968] proved that for large N, (29) holds for any short-memory time series model. *Siddiqui* [1976] demonstrated that for any model with a summable TACVF,

$$a = \left(\frac{\pi}{2\gamma_0} \sum_{i=-\infty}^{\infty} \gamma_i \right)^{1/2} \tag{30}$$

It has been argued by some authors that because short-memory models, such as the Arma processes, imply a limiting value of K equal to 0.5 and since the observed K in annual geophysical time series is about 0.7, short-memory models are not appropriate models for synthetic streamflow generation. It should therefore be emphasized that asymptotic results are only relevant in that they provide an approximation to the exact results for the true (finite) series length.

Anis and Lloyd [1976] showed that (28) also holds exactly for symmetrically correlated normal summands. But such a time series has a long memory, since its TACVF is not summable. Because (28) is also valid for short-memory NID random variables, this fact provides a counterexample to the claim of some researchers that long-memory models are a necessary explanation of the Hurst phenomenon. Conversely, *Klemes* [1974] has shown that zero-memory nonstationary models could produce the Hurst phenomenon. By simulation experiments with white noise, he varied the mean level in different manners and showed how K increased in value due to this type of nonstationarity. Klemes also demonstrated by simulation that random walks with one absorbing barrier, which often arise in natural storage systems, could cause the RAR to have certain properties related to the Hurst phenomenon.

Hurst [1957] was the first scientist to suggest that a nonstationary model in which the mean of the series was subject to random changes could account for higher values of the Hurst coefficient K and hence the Hurst phenomenon. Similar models have been studied by *Klemes* [1974] and *Potter* [1976]. However, such processes do not seem to be appealing for use in synthetic streamflow generation, since they would be difficult to fit properly to a given historical time series. *Hipel et al.* [1975, 1977*b*] have described how known changes in the mean level of a river flow time series can be modeled by intervention analysis.

Matalas and Huzzen [1967] performed statistical experiments to determine whether K is preserved by Markov models. For values of the lag 1 autocorrelation coefficient ρ_1 ranging from 0 to 0.9 they calculated the $E(K)$ based upon 10^4 simulations for particular values of N and ρ_1. For values of N and ρ_1, compatible with what occurs in annual river flows if those flows are assumed Markov, they found K to have an average of about 0.7. Because a mean of approximately 0.7 for K occurs in natural time series, they implied that perhaps the small sample properties of K are preserved by a Markov model. Nevertheless, a later simulation study of *Wallis and Matalas* [1970] suggested that the observed sample lag 1 autocorrelations for flows in the Potomac River basin were too low for a first-order AR process adequately to preserve the Hurst K. However, a Markov model may not necessarily be the best short-memory model to fit to a given time series. Rather, it is recommended to select the proper Arma model by adhering to the identification, estimation, and diagnostic check stages of model construction [*Box and Jenkins,* 1970; *Hipel et al.,* 1977*a*; *McLeod et al.,* 1977; *McLeod,* 1977, 1978; *McLeod and Hipel,* 1978*a*]. In some cases the appropriate model may indeed be a Markov model. *Hipel and McLeod* [1978*a*] demonstrate that for 23 geophysical time series ranging in length from $N = 96$ to $N = 1164$ properly fit Arma models do adequately preserve K.

Several other authors have also suggested that short-memory models may preserve K. *Gomide* [1975] has completed further simulation studies of the RAR for Markov models. *O'Connell* [1974*a, b*] advocated employing an Arma model with one AR and one MA term (Arma (1, 1)) to approximate

the long-memory FGN and thereby perhaps to preserve K. To accomplish this, the AR parameter must have a value close to unity, so that the autocorrelation function (ACF) of the process will attenuate slowly and hence approximate the FGN TACF. In practice, this approach may not be viable. The proper Arma model that is fit to the data may not be Arma (1, 1), and even if it is Arma (1, 1), an efficient MLE of the parameters may not produce an estimate of the AR parameter that is close to 1. This parameter estimation problem is acknowledged by *O'Connell* [1976]. In addition, it is no longer necessary to approximate FGN by a short-memory model such as an Arma (1, 1) model because as is shown later in this paper it is now possible to simulate FGN exactly.

Long-memory models. A long-memory model known as FGN was developed perhaps in order to explain the Hurst phenomenon. The link with Hurst's law is the parameter H in FGN that is often estimated by the Hurst coefficient K in (17). The FGN model was first proposed by *Mandelbrot* [1965], and a mathematical derivation was given by *Mandelbrot and Van Ness* [1968] and *Mandelbrot and Wallis* [1969*c*]. The literature concerning the FGN model has recently been summarized by *Wallis and O'Connell* [1973], *O'Connell* [1974*b*, chapter 2], and *Lawrance and Kottegoda* [1977]. Consequently, only the main historical points of practical interest will be discussed. This will be followed in the next section by a presentation of new advancements in FGN modeling that include exact simulation, efficient parameter estimation, and model diagnostic checking.

In the development of FGN processes, *Mandelbrot* [1965] considered a continuous time process $B_H(t)$ that satisfied the self-similarity property such that for all τ and $\epsilon > 0$, $B_H(t + \tau) - B_H(t)$ has exactly the same distribution as $[B_H(t + \tau\epsilon) - B_H(t)]/\epsilon^H$. It can be shown that the sequential range of $B_H(t)$ will increase proportionally to N^H, where the sequential range is defined by

$$\max_{t<r<t+N} B_H(r) - \min_{t<r<t+N} B_H(r) \tag{31}$$

where t is continuous time and H is the model parameter. When the process $B_H(t)$ is Gaussian, it is called fractional Brownian motion. Discrete time fractional Gaussian noise is defined for discrete time t by the increments

$$z_t = B_H(t + 1) - B_H(t) \tag{32}$$

FGN is what *Mandelbrot and Wallis* [1969*c*] consider to be a model of Hurst's geophysical time series.

Mandelbrot and Van Ness [1968] and *Mandelbrot and Wallis* [1969*a, b, c*] have derived a number of properties of FGN. First, the parameter H must satisfy the inequality $0 < H < 1$. The sample mean and variance of FGN are consistent estimators of the true mean and variance, and FGN is covariance stationary. The expected values of the crude and adjusted ranges for FGN are the asymptotic relationships

$$E(R_N) = a_H N^H \qquad 0 < H < 1 \tag{33}$$

and

$$E(R_N^*) = b_H N^H \qquad 0 < H < 1 \tag{34}$$

where a_H and b_H are coefficients that do not depend on N. It can also be shown that for large N [*Rao*, 1973, p. 122],

$$E(\bar{R}_N^*) = aN^H \tag{35}$$

Although the above asymptotic formulae are correct mathematically, they may possess limitations with respect to modeling Hurst's findings. Of foremost importance is the fact that Hurst examined $\bar{R}_N^*$ for small N and not the asymptotic expected values of R_N, R_N^*, and $\bar{R}_N^*$. Behavior of any of the range statistics for large N does not necessarily infer the structure of $\bar{R}_N^*$ for small and moderate N. Even though (33)–(35) are asymptotically valid, in reality the Hurst coefficient is a function of N and is not a constant as is the parameter H in FGN. For example, as is shown by simulation experiments for NID random variables in Table 6, the expected value of the Hurst coefficient K is significantly larger than 0.5 for small N. A sequence of NID random variables is equivalent to a FGN process with $H = \frac{1}{2}$.

The TACF at lag k of FGN is given by

$$\rho_k = \tfrac{1}{2}[(k + 1)^{2H} - 2k^{2H} + (k - 1)^{2H}] \qquad 0 < H < 1 \quad k \geq 1 \tag{36}$$

For large lags, (33) may be approximated by

$$\rho_k = H(2H - 1)k^{2H-2} \tag{37}$$

An examination of (36) and (37) reveals that $\rho_k \to 0$ as $k \to \infty$, but ρ_k is not summable if $\frac{1}{2} < H < 1$. Therefore for $\frac{1}{2} < H < 1$, FGN is a long-memory process. When $0 < H \leq \frac{1}{2}$, FGN constitutes a short-memory model.

For many geophysical phenomena the estimates for H are greater than $\frac{1}{2}$ but less than 1. Because FGN is summable for H in this range, the statistical effect of past events on present behavior attenuates very slowly. Therefore long-term persistence, as described by the TACF, is synonymous with $\frac{1}{2} < H < 1$. Some hydrologists claim that the form of the TACF for $\frac{1}{2} < H < 1$ is explained by the physical existence of an extremely long memory in hydrologic and other processes. But, as was pointed out by *Klemes* [1974], making inferences about physical features of a process based on operational models can be not only inaccurate but also misleading. Klemes correctly states that '. . . it must be remembered that the mathematical definition of FGN did not arise as a result of the physical or dynamic properties of geophysical and other processes but from a desire to describe an observed geometric pattern of historic time series mathematically . . . Thus FGN is an operational, not a physically founded model.' Klemes demonstrates that the Hurst phenomenon could be due to zero-memory nonstationary models and also specific types of storage systems. However, although physical interpretations that use operational models should be formulated and interpreted with caution, one criterion that is essential is that the statistical properties of any historical time series be incorporated properly into the stochastic model.

The appropriateness of long-memory processes for modeling annual river flow and other types of natural time series has been questioned previously by various hydrologists [*Scheidegger*, 1970; *Klemes*, 1974]. Moreover, later in this paper it is shown that the FGN model can fail to provide an adequate statistical fit to historical annual river flows.

IMPROVED FGN TECHNIQUES

The FGN model for a time series z_t, $t = 1, 2, \cdots$, can be specified in terms of the three parameters μ, γ_0, and H, where $E(z_t) = \mu$, Var $(z_t) = \gamma_0$, and the TACF of z_t is given by (36). From these specifications, improved simulation and estimation procedures can be developed. Complete Fortran computer algorithms for these new methods are given by *Hipel and McLeod* [1978*b*].

When determining a long-memory or a short-memory model or in general any type of stochastic process for modeling a given data set, it is recommended to adhere to the three stages of model development [*Box and Tiao,* 1973; *Box and Jenkins,* 1970; *Hipel et al.,* 1977*a*; *McLeod et al.,* 1977]. The first step consists of identifying, or choosing, the type of model to fit to the time series. If circumstances warrant the employment of a FGN process, then at the estimation stage, efficient MLE of the model parameters can be procured by using the technique developed in this section. It is also shown how the model residuals of FGN can be calculated after the model parameters have been estimated. If diagnostic checks of the residuals reveal that modeling assumptions such as residual whiteness, normality, and homoscedasticity (i.e., constant variance) are not satisfied, then appropriate action can be taken. For example, a transformation of the data prior to fitting a FGN process may rectify certain anomalies in the residuals. In some cases a short-memory model such as an Arma process may provide a better statistical fit while at the same time preserve important historical statistics such as the RAR. The AIC [*Akaike,* 1974] is recommended as a means of selecting the best model from a set of tentative models that may consist of both short-memory and long-memory processes.

Simulation of FGN

Historically, researchers have not developed an exact technique for simulating FGN. Instead short-memory approximations to FGN models have been utilized to generate synthetic traces. The methods used for obtaining approximate realizations of FGN include (1) type 1 [*Mandelbrot and Wallis,* 1969*c*], (2) type 2 [*Mandelbrot and Wallis,* 1969*c*], (3) fast FGN [*Mandelbrot,* 1971], (4) filtered FGN [*Matalas and Wallis,* 1971], (5) Arma (1, 1) [*O'Connell,* 1974*a*, *b*], (6) broken line [*Rodríguez-Iturbe et al.,* 1972; *Mejia et al.,* 1972; *Garcia et al.,* 1972; *Mandelbrot,* 1972], and (7) Arma-Markov [*Lettenmaier and Burges,* 1977].

Various papers have been written that include surveys and appraisals of one or more of the short-memory approximations to FGN [see *Lawrance and Kottegoda*, 1977; *Lettenmaier and Burges*, 1977; *O'Connell,* 1974*b*; *Wallis and O'Connell,* 1973]. Although the underlying drawback of all these approximate processes is that the simulated data does not lie outside the Brownian domain (see *Mandelbrot and Wallis* [1968] for a definition of Brownian domain), additional handicaps of some of the models have also been cited in the literature. For instance, *Lawrance and Kottegoda* [1977] mention that the lack of a suitable estimation procedure for the parameters of a broken line process is the greatest deterrent to the utilization of that model by hydrologists.

When generating synthetic traces from a short-memory approximation to FGN or any other type of stochastic model, proper simulation procedures should be adhered to. If more than one simulated time series from a certain model is needed, then it would be improper to first simulate one long synthetic time series and then to subdivide this longer trace into the required number of shorter time series. Rather, it would be more efficient to generate the shorter series independently, so that the resulting estimates from each of the shorter series would be statistically independent. Furthermore, the standard errors of the particular parameters being estimated by the simulation study can be calculated if the estimates are statistically independent, but if they are correlated, the standard errors are not easily estimated.

Instead of the employment of short-memory approximations for simulating FGN it is possible to generate exact realizations of FGN. Suppose that a FGN series $z_1, z_2, \cdots, z_N$ with parameters μ, γ_0, and H is to be simulated. First, by utilizing an appropriate standard method generate a Gaussian white noise sequence $e_1, e_2, \cdots, e_N$ that is NID (0, 1). Next calculate the $N \times N$ correlation matrix,

$$\mathbf{C}_N(H) = [\rho_{|i-j|}] \tag{38}$$

where ρ_0 equals 1 and ρ_k is calculated from (36) for $k \geq 1$. Then the Cholesky decomposition [*Healy,* 1968] of $\mathbf{C}_N(H)$ is determined in such a way that

$$\mathbf{C}_N(H) = \mathbf{M}\mathbf{M}^T \tag{39}$$

where $\mathbf{M}$ equals (m_{ij}) is the $N \times N$ lower triangular matrix. Exact realizations of FGN are calculated from

$$z_t = \mu + \left(\sum_{i=1}^{t} m_{ti}\, e_i \right)(\gamma_0)^{1/2} \tag{40}$$

for $t = 1, 2, \cdots, N$ and for $0 < H < 1$, where z_t is the FGN time series value that is $N(\mu, \gamma_0)$. If the model parameter H is in the range $0.5 < H < 1$, then the synthesized data will lie outside the Brownian domain.

The computer algorithm for exactly simulating FGN is listed in standard Fortran by *Hipel and McLeod* [1978*b*]. This algorithm requires only about $\frac{1}{2}N(N + 2)$ storage locations to simulate a FGN series of length N. Thus a modest requirement of about 5000 words is required to handle a series of length 100.

Maximum Likelihood Estimation

In addition to the mean and variance, an estimate of the parameter H forms the only link that a FGN model has with the real world as represented by the historical data. Previously, various estimates for H have been formulated. Some researchers employ K in (17) as an estimate of H. *Wallis and Matalas* [1970] recommend the G Hurst statistics as an estimate of H. Unfortunately, little is known about the theoretical distribution of this estimate, and the G Hurst statistic in effect constitutes only an ad hoc method of calculating H. *Young and Jettmar* [1976, p. 830, equation (4)] suggest a moment estimate for H based on an estimate of the historical ACF at lag 1 and (36). They also develop a least squares estimate for H that is formulated by using the sample ACF and (36) [*Young and Jettmar,* 1976, p. 831, equation 6]. However, *McLeod and Hipel* [1978*a*] question the theoretical basis and efficiency of Young and Jettmar's least squares estimate for H.

An alternative approach to estimating the parameters of a FGN model is to employ the method of ML. The ML estimation procedure is widely used for the estimation of parametric models, since it often yields the most efficient estimates. *Dunsmuir and Hannan* [1976] have shown that the MLE of the parameters of time series models often yield optimal estimates under very general conditions which include the FGN model as a special case.

Given a historical time series $z_1, z_2, \cdots, z_N$ the log likelihood of μ, γ_0, and H in the FGN model is

$$\log L(\mu, \gamma_0, H) = -\tfrac{1}{2} \log |\mathbf{C}_N(H)| - (2\gamma_0)^{-1} S(\mu, H) - (N/2) \log \gamma_0 \tag{41}$$

where $\mathbf{C}_N(H)$ is the correlation matrix given by (38). The function $S(\mu, H)$ in (41) is determined by

$$S(\mu, H) = (\mathbf{z} - \mu\mathbf{1})^T [\mathbf{C}_N(H)]^{-1} (\mathbf{z} - \mu\mathbf{1}) \tag{42}$$

TABLE 3. Estimated Statistics for the Annual River Flows

Data Set	$\hat{H}$*	$GH(10)$	K
Mstouis	0.674	0.580	0.648
	(0.082)		
Neumunas	0.591	0.520	0.660
	(0.067)		
Danube	0.548	0.560	0.633
	(0.063)		
Rhine	0.510	0.592	0.614
	(0.058)		
Ogden	0.949	0.868	0.894
	(0.047)		
Gota	0.839	0.523	0.689
	(0.073)		

*The parenthetical values are standard deviations.

each of these time series the MLE of H in the FGN model and its standard error are calculated by using the Fortran computer programs given by *Hipel and McLeod* [1978*b*]. The calculations require only about 11 min of computer time on a Honeywell 6060 Computer System. Table 3 lists the MLE and standard errors (in parentheses) of H and also the Hurst K and $GH(10)$ estimates for each of the time series.

In Table 3 notice the difference between the three estimates of the FGN parameter H for each of the data sets. For instance, $\hat{H}$ for the Gota River has a magnitude of 0.839 with a corresponding standard error of 0.073. Both the $GH(10)$ and K estimates for the Gota River are more than 2 times the standard error less than the MLE of H.

The parameter estimates for the proper Arma models that are fit to the time series in Table 2 are given by *Hipel and McLeod* [1978*a*, Table 3]. Both the Danube River and the Rhine River time series are simply white noise. If a time series is NID, the theoretical value of H for a FGN model is 0.5. For both the Danube River and the Rhine River, Table 3 reveals that the MLE of H is closer to 0.5 than either the $GH(10)$ or the K estimate. In addition, for each of the two data sets, $\hat{H}$ is easily within 1 standard error of 0.5.

In order to determine whether a short-memory or a long-memory model should be selected for each of the six time series, the AIC can be implemented. Table 4 lists the values of the AIC for the FGN models by using $\hat{H}$ and the best fitting Box-Jenkins Arma model. For each of the six cases the AIC for the Arma model has a magnitude less than that for the FGN model. Therefore, on the basis of a combination of best statistical fit and model parsimony the Arma model should be chosen in preference to the FGN process for the time series considered.

The Gota River is instructive for portraying possible problems that may arise when using FGN models in practice, since it appears that no FGN model can give an adequate fit to this time series. After a FGN model has been fit to a given data set it is recommended to implement appropriate diagnostic checks

TABLE 4. AIC of the Fitted FGN and Arma Models

Data Set	FGN Models	Arma Models
Mstouis	1400.0	1395.8
Neumunas	1207.5	1198.2
Danube	1666.7	1389.0
Rhine	1531.8	1529.8
Ogden	1176.9	1172.1
Gota	1350.6	1331.0

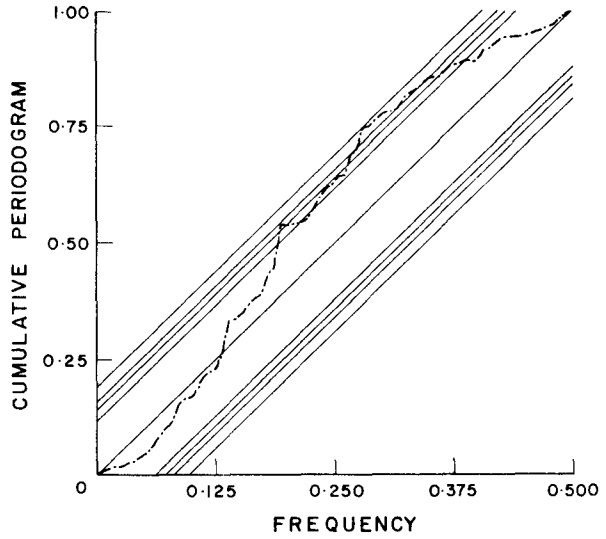

Fig. 2. Gota River residual cumulative periodogram for the FGN model using $\hat{H}$.

for testing model adequacy. It is of utmost importance that the residuals of FGN given by (51) be white noise. Accordingly, plots of the cumulative periodogram for the residuals of the FGN models for the Gota River obtained by using $\hat{H}$, $GH(10)$, and K are displayed in Figures 2–4, respectively. The 1%, 5%, 10%, and 25% significance levels are indicated on the plots. As is shown in the figures, the cumulative periodogram test is significant in all three cases at the 1% level, although the departure from whiteness is not as great for the FGN model when using $\hat{H}$ as it is for the other two cases. Therefore the whiteness diagnostic checks indicate that because of the dependence of the model residuals the FGN processes provide a poor statistical fit to the given data. Hence it would be advisable to consider another type of process to model the annual river flows of the Gota River.

When selecting a process to describe a given time series, it is highly desirable that important historical statistics such as the ACF at various lags (especially at low lags for nonseasonal models) be preserved. The inability of the three FGN models

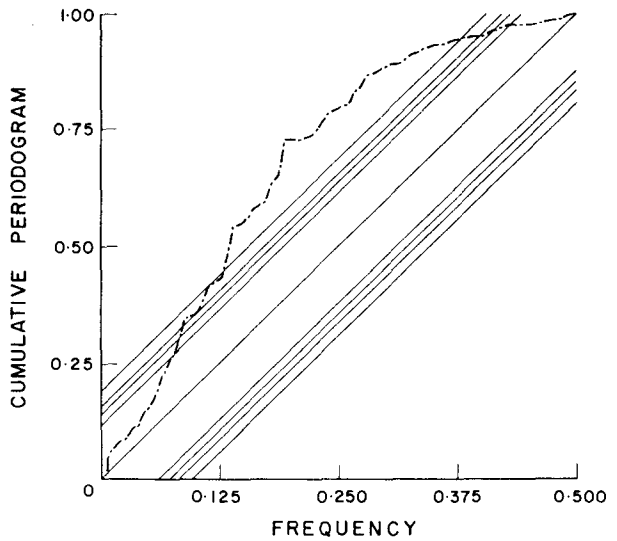

Fig. 3. Gota River residual cumulative periodogram for the FGN model using $GH(10)$.

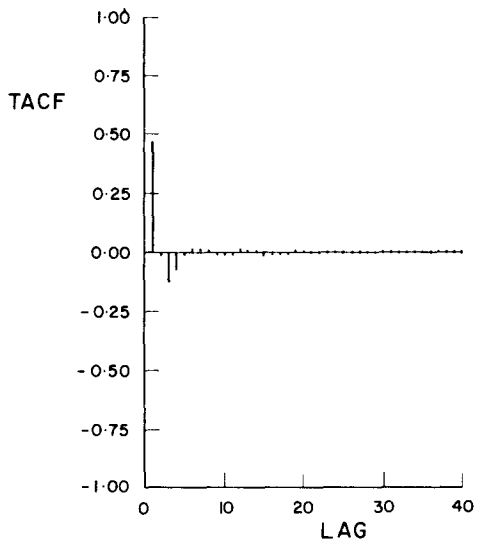

Fig. 8. TACF of the Gota River for the Arma (2, 0) model.

processes preserve the RAR or equivalently K. This procedure could also be adopted for statistics such as various lags of the ACF to show quantitatively whether these statistics are preserved by the models.

The inability of a FGN process to preserve the ACF and perhaps other historical statistics in some practical applications could be due to the inherent mathematical structure and underlying properties that were discussed previously. Another obvious drawback of FGN is the dependence of the model on only a few parameters. In addition to the mean and variance an estimate of the parameter H forms the only actual link between the theoretical model and the real world as represented by the data. This renders FGN processes highly inflexible. On the other hand, in Box-Jenkins modeling the form of the model is tailored specifically to fit a given set of data. At the identification stage the general structure of the data is determined by observing the shape of the ACF and other graphs such as the partial ACF (PACF), the inverse ACF (IACF), and the inverse PACF (IPACF) [*Hipel et al.*, 1977*a*; *McLeod et al.*, 1977]. An appropriate number of AR and MA parameters are chosen in order that the chosen Arma model fits the data as closely as possible. Rigorous checks are performed to insure that the white noise component of the model is not correlated. If all the modeling assumptions are satisfied, this guarantees that important historical statistics such as the ACF, the RAR, and K will be preserved reasonably well by the model.

Simulation Studies

When studying statistics such as the RAR and K, information is required regarding first-, second-, and perhaps higher-order moments of the statistics. In general, it would be most advantageous to know the exact distribution of the statistic under study. Three approaches are available to obtain knowledge regarding the mathematical properties of a specified statistic. One method is to derive an exact analytical expression for the moments and perhaps the distribution of the statistic. Except for special cases of the lower-order moments of a statistic, this precise procedure is often analytically intractable. Only recently, *Anis and Lloyd* [1976] were able to derive in (28) the exact expression for the expected value of the RAR for NID summands.

A second approach is to develop asymptotic formulae for the distributional properties of a given statistic. This approximate procedure may yield results that are useful in certain situations, while in other circumstances the output may suffer from lack of accuracy, especially for small N. *Feller* [1951], for example, proved an asymptotic relationship that is valid for the expected value of the adjusted range and also the RAR of IID random variables (see (18) and (19), respectively). *Siddiqui* [1976] derived asymptotic expressions for calculating the expected value of the RAR for any short-memory process.

In the third approach, simulation is used to determine as accurately as desired the distributional attributes of a given statistic. *Hipel and McLeod* [1978*a*] utilize Monte Carlo procedures to obtain the empirical distribution of the RAR and K. Although some researchers may argue that simulation may be relatively costly with respect to computer usage, the fact of the matter is that answers are needed now to help solve present-day engineering problems. In addition, because of the vast mathematical complexity that is often required to prove exact analytical solutions, simulation results may help to economize academic endeavors by delineating the more promising avenues of research that could also be scrutinized analytically. Finally, it should be borne in mind that in comparison with an exact analytical solution, simulation provides a straightforward but equally correct resolution to the problem of the distributional characteristics of a particular statistic.

The simulation investigations of this section deal primarily with the estimated mean and variance of a certain statistic. Suppose that $\bar{N}$ independent simulations of a time series $z_1, z_2, \cdots, z_N$ are obtained and that a statistic $T = T(z_1, z_2, \cdots, z_N)$ is calculated in each simulated series. The empirical mean of T is then given by

$$\bar{T} = \frac{1}{\bar{N}} \sum_{i=1}^{\bar{N}} T_i \tag{52}$$

where T_i is the value of T in the ith simulation. If each successive realization of the sequence $z_1, z_2, \cdots, z_N$ is independent of previous realizations so that the T_i are statistically independent, then the variance of T can be estimated by

$$V_T = \frac{1}{\bar{N} - 1} \sum_{i=1}^{\bar{N}} (T_i - \bar{T})^2 \tag{53}$$

By the central limit theorem, $\bar{T}$ will be distributed very nearly normally with mean equal to $E(\bar{T})$ and with variance approximately equal to $V_T/\bar{N}$. Thus the standard deviation and confidence intervals for the expected value being estimated are readily obtained.

If $\bar{N}$ white noise series of length N are being simulated, then it is correct to simulate a single time series of length $\bar{N}N$ and then subdivide it into $\bar{N}$ series with N values in each series. However, if a correlated series is being simulated, the aforementioned procedure should not be followed. For instance, if $\bar{N}$ FGN series with $0.5 < H < 1$ are being formulated by first generating a long series of length $\bar{N}N$ and then subdividing this into $\bar{N}$ subsequences of length N, then the resulting T_i will in general be correlated. Therefore the resulting estimate for $E(T)$ in (52) will be less precise (i.e., have larger variance), and the estimate of the variance of T in (53) will be underestimated, so that correct standard deviations and confidence intervals for $E(T)$ will not be available.

Simulation of Independent Summands

The rescaled adjusted range. *Mandelbrot and Wallis* [1969*e*] reported simulation experiments which indicated that the ex-

TABLE 5. Expected Value of the RAR for Some IID Summands

	Analytical Results		Simulation Results*			
N	*Anis and Lloyd* [1976]	*Feller* [1951]	Normal	Gamma	Stable	Cauchy
5	1.9274	2.8025	1.9273	1.9851	1.9264	1.9506
			(0.0027)	(0.0018)	(0.0022)	(0.0026)
10	3.0233	3.9633	3.0302	3.0330	2.9699	3.0556
			(0.0060)	(0.0039)	(0.0047)	(0.0056)
15	3.8812	4.8541	3.8826	3.8356	3.7571	3.8987
			(0.0084)	(0.0056)	(0.0064)	(0.0079)
20	4.6111	5.6050	4.6047	4.5141	4.4408	4.6214
			(0.0100)	(0.0071)	(0.0075)	(0.0098)
25	5.2576	6.2666	5.2540	5.1213	5.0044	5.2889
			(0.0116)	(0.0085)	(0.0088)	(0.0115)
30	5.8443	6.8647	5.8770	5.6709	5.5681	5.8767
			(0.0131)	(0.0097)	(0.0098)	(0.0130)
35	6.3851	7.4147	6.4214	6.1707	6.0090	6.3974
			(0.0145)	(0.0109)	(0.0106)	(0.0143)
40	6.8895	7.9267	6.8920	6.6605	6.5037	6.9075
			(0.0158)	(0.0121)	(0.0118)	(0.0155)
45	7.3640	8.4075	7.3595	7.0938	6.9010	7.3934
			(0.0169)	(0.0132)	(0.0125)	(0.0166)
50	7.8133	8.8623	7.7785	7.5012	7.3184	7.8540
			(0.0180)	(0.0141)	(0.0132)	(0.0178)
60	8.6502	9.7081	8.6246	8.3061	8.0670	8.6263
			(0.0198)	(0.0159)	(0.0148)	(0.0195)
70	9.4210	10.4860	9.4453	9.0632	8.7242	9.4454
			(0.0215)	(0.0178)	(0.0158)	(0.0211)
80	10.1392	11.2100	10.1349	9.7327	9.3732	10.1336
			(0.0233)	(0.0194)	(0.0172)	(0.0232)
90	10.8143	11.8900	10.8208	10.4068	9.9544	10.8857
			(0.0248)	(0.0209)	(0.0183)	(0.0248)
100	11.4533	12.5331	11.4775	10.9769	10.5593	11.4546
			(0.0262)	(0.0224)	(0.0196)	(0.0258)
125	12.9243	14.0125	12.9617	12.4280	11.8353	12.9619
			(0.0299)	(0.0255)	(0.0220)	(0.0292)
150	14.2556	15.3499	14.1956	13.6864	13.0622	14.2636
			(0.0323)	(0.0285)	(0.0240)	(0.0323)
175	15.4806	16.5798	15.4198	14.8752	14.1069	15.4971
			(0.0349)	(0.0315)	(0.0261)	(0.0354)
200	16.6214	17.7245	16.5938	15.9992	15.1381	16.6259
			(0.0376)	(0.0337)	(0.0281)	(0.0376)

*The parenthetical values are standard deviations.

pected value of the RAR for IID summands is virtually independent of the underlying distribution. However, as was pointed out by *Taqqu* [1970], the simulation study of *Mandelbrot and Wallis* [1969e] contained a serious programing error in the calculation of the RAR. Accordingly, another study of the robustness of the expected value of the RAR with respect to the underlying distribution is required.

A simulation study is performed for various types of white noise series varying in length from $N = 5$ to $N = 200$. For each value of N the number of series of length N that are generated is $\bar{N} = 10{,}000$. The expected values of the RAR are determined by using (52) for the following independent summands: (1) normal, (2) gamma with shape parameter 0.1, (3) symmetric stable with characteristic exponent $\alpha = 1.3$, and (4) cauchy. In Table 5 are listed the simulation results for $E(\bar{R}_N^*)$ at specific values of N for the aforementioned summands. The standard deviations of the estimated values of $E(\bar{R}_N^*)$ are determined by using the square root of (53) and are given in parentheses below the estimates in Table 5. The exact values of $E(\bar{R}_N^*)$ for NID random variables are calculated by using the formula of *Anis and Lloyd* [1976] that is written in (28). A comparison of columns 2 and 4–7 reveals that the expected value of the RAR is indeed rather insensitive to the underlying distribution for the values of N that are considered. Even for cauchy summands the expected value and variance of the RAR are quite similar to the NID case. The asymptotic results of *Feller* [1951] for $E(\bar{R}_N^*)$ of IID summands are determined by using (19) and are tabulated in Table 5. A perusal of the asymptotic and other entries in the table discloses that the approximation given by Feller's results improves with increasing N.

Anis and Lloyd [1975] developed analytical formulae for the expected value of the crude and adjusted ranges of independent gamma random variables. For highly skewed independent gamma summands the local Hurst coefficient for the crude and adjusted ranges possessed values greater than 0.5 for N less than 1000. However, the results of Table 5 indicate that the expected values of the RAR for IID summands are approximately independent of the underlying distribution even if that distribution is gamma. Therefore as was confirmed by *O'Connell* [1976], *Anis and Lloyd*'s [1975] results do not hold for the RAR. In addition, Hurst studied K for the RAR and not the Anis and Lloyd local Hurst coefficient for the crude and adjusted ranges.

The Hurst coefficient. As was mentioned previously, the Hurst statistic of primary import is the RAR. Nevertheless, because the Hurst coefficient K has been extensively investi-

TABLE 6. Hurst Coefficients for NID Summands

N	K'	$E(K)$*	YH'
5	0.7161	0.7032	0.3375
		(0.0016)	
10	0.6874	0.6750	0.4315
		(0.0013)	
15	0.6731	0.6629	0.4591
		(0.0011)	
20	0.6638	0.6540	0.4725
		(0.0010)	
25	0.6571	0.6469	0.4805
		(0.0009)	
30	0.6519	0.6420	0.4859
		(0.0008)	
35	0.6477	0.6385	0.4897
		(0.0008)	
40	0.6442	0.6365	0.4926
		(0.0007)	
45	0.6413	0.6335	0.4948
		(0.0007)	
50	0.6387	0.6305	0.4967
		(0.0007)	
60	0.6344	0.6270	0.4994
		(0.0007)	
70	0.6309	0.6235	0.5014
		(0.0006)	
80	0.6279	0.6213	0.5029
		(0.0006)	
90	0.6254	0.6186	0.5040
		(0.0006)	
100	0.6233	0.6156	0.5049
		(0.0006)	
125	0.6189	0.6129	0.5066
		(0.0005)	
150	0.6154	0.6100	0.5078
		(0.0005)	
175	0.6127	0.6070	0.5086
		(0.0005)	
200	0.6103	0.6051	0.5092
		(0.0005)	

*The parenthetical values are standard deviations.

gated during the past quarter of a century, this fact may insure the survival of K as an important hydrological statistic for some time to come. Therefore some statistical properties of K and other exponents are investigated.

First, it should be noted that because of the research results of *Anis and Lloyd* [1976] in (28), K can be evaluated analytically for NID summands. Let K' be the Hurst coefficient calculated by using

$$K' = \log E(\bar{R}_N^*)/(\log N - \log 2) \tag{54}$$

where $E(\bar{R}_N^*)$ is determined exactly by using (28). It follows from Jensen's inequality [*Rao*, 1973, p. 57] that for finite N,

$$E(K) < K' \tag{55}$$

In Table 6 the magnitudes of K' from (54) are listed for the length of series N ranging from 5 to 200. When 10,000 series are generated for NID random variables for each N, then the expected value of K can be estimated by utilizing (52), while the standard deviation of $E(K)$ can be calculated by using the square root of (53). In Table 6 the estimated values of $E(K)$ for various time series lengths are cataloged. The standard deviations of the estimates are contained in the parentheses below each estimate. A comparison of columns 2 and 3 in Table 6 demonstrates that the inequality in (55) is valid. However, the difference between $E(K)$ and K' is negligible. Therefore (54) provides a viable means for estimating the expected value of K for NID summands. In addition, the Hurst coefficient K is obviously a function of the sample size, and for increasing N the coefficient K attenuates toward its asymptotic value of 0.5. However, for small and moderate values of N, K is significantly larger than 0.5.

The coefficient K constitutes one method of estimating the generalized Hurst coefficient h in (15). Another approach is to evaluate h by using the estimate YH of *Gomide* [1975] that is given in (20). By taking logarithms of (20) an explicit expression for YH is

$$YH = (\log \bar{R}_N^* - \log 1.2533)/\log N \tag{56}$$

Although the expected value of YH could be determined from simulation experiments, an alternative analytical procedure is to substitute $E(\bar{R}_N^*)$ from (28) for $\bar{R}_N^*$ in (56) and then to estimate YH by YH' by using (56). In Table 6 the values of YH' are tabulated for different time series lengths. It is obvious that YH' is a function of the sample size and that YH' provides a closer approximation to the limiting value of 0.5 than does K.

Simulation of Correlated Summands

Long-memory models. By utilizing (40) and the accompanying Fortran computer algorithms of *Hipel and McLeod*

TABLE 7. Expected Value of the RAR for FGN Models

	FGN Model*	
N	$H = 0.7$	$H = 0.9$
5	1.9682	2.0100
	(0.0026)	(0.0025)
10	3.2716	3.5031
	(0.0062)	(0.0061)
15	4.3946	4.8751
	(0.0091)	(0.0094)
20	5.3972	6.1579
	(0.0116)	(0.0125)
25	6.3351	7.4051
	(0.0141)	(0.0155)
30	7.2066	8.6032
	(0.0165)	(0.0187)
35	8.0515	9.7839
	(0.0188)	(0.0216)
40	8.8767	10.9431
	(0.0205)	(0.0241)
45	9.6650	12.0926
	(0.0227)	(0.0271)
50	10.4007	13.2284
	(0.0247)	(0.0298)
60	11.8233	15.3575
	(0.0280)	(0.0352)
70	13.2003	17.4965
	(0.0322)	(0.0413)
80	14.5205	19.5945
	(0.0356)	(0.0461)
90	15.7709	21.6075
	(0.0389)	(0.0518)
100	16.9241	23.5818
	(0.0420)	(0.0573)
125	19.8877	28.5197
	(0.0494)	(0.0700)
150	22.6178	33.2646
	(0.0571)	(0.0831)
175	25.2291	38.0410
	(0.0638)	(0.0964)
200	27.7601	42.6710
	(0.0701)	(0.1080)

*The parenthetical values are standard deviations.

TABLE 8. Expected Value of the RAR for a Markov Model With $\phi_1 = 0.3$

N	$E(\bar{R}_N^*)$ Asymptotic	$E(\bar{R}_N^*)$ Simulated*
5	3.8192	1.9875 (0.0026)
10	5.4011	3.3410 (0.0062)
15	6.6150	4.4633 (0.0089)
20	7.6383	5.4261 (0.0114)
25	8.5390	6.2853 (0.0135)
30	9.3550	7.0666 (0.0156)
35	10.1045	7.7976 (0.0175)
40	10.8022	8.5022 (0.0188)
45	11.4575	9.1493 (0.0205)
50	12.0772	9.7347 (0.0221)
60	13.2299	10.8709 (0.0242)
70	14.2900	11.9207 (0.0273)
80	15.2766	12.9177 (0.0296)
90	16.2033	13.8181 (0.0317)
100	17.0798	14.6243 (0.0335)
125	19.0958	16.6970 (0.0380)
150	20.9184	18.5288 (0.0424)
175	22.5944	20.1758 (0.0459)
200	24.1545	21.7339 (0.0491)

*The parenthetical values are standard deviations.

[1978b] it is possible to simulate exactly synthetic traces of FGN. Because only short-memory approximations to FGN processes were previously available for simulation purposes, the exact method should prove useful for checking former FGN simulation studies and also for exploring new avenues of research for long-memory models. Of particular importance are Monte Carlo studies to investigate the statistical properties of FGN processes. Consider, for example, the behavior of the RAR for FGN models. For time series varying in length from $N = 5$ to $N = 200$ a total of 10,000 simulated sequences are generated for each value of N. Because the RAR statistic is not a function of the mean and variance of a FGN process, it is convenient to assign the mean a value of zero and the variance a magnitude of 1 when performing the simulations by using (40). By utilizing (52) and (53) the expected values of the RAR and variances, respectively, are calculated. Table 7 records the estimate of $E(\bar{R}_N^*)$ and the corresponding standard deviations in brackets for FGN models with $H = 0.7$ and 0.9. From an inspection of the entries in Table 7 it is obvious that $E(\bar{R}_N^*)$ increases in magnitude for larger N. Furthermore, at a given value of N the expected value of the RAR is greater for a FGN model with $H = 0.9$ than it is for a FGN process with $H = 0.7$.

Short-memory models. *McLeod and Hipel* [1978b] have recently developed improved simulation procedures for generating synthetic traces for Box-Jenkins models. These techniques have been called WASIM1 (Waterloo simulation procedure 1) and WASIM2 (Waterloo simulation procedure 2). The standard Fortran programs for WASIM1 and WASIM2, along with other supporting subroutines and documentation, are listed in the microfiche edition of the paper by *McLeod and Hipel* [1978b]. When either WASIM1 or WASIM2 is employed, random realizations of the process under consideration are used as starting values. Since fixed initial values are not utilized, systematic bias is avoided in the generated data.

As a typical example of a short-memory process, consider the Markov model given by

$$z_t = \phi_1 z_{t-1} + a_t \tag{57}$$

where t equals 1, 2, $\cdots$, N, ϕ_1 is the AR parameter, and a_t is the white noise that is NID $(0, \sigma_a^2)$. By using WASIM2 a total of 10,000 synthetic sequences are generated for specific values of N for Markov processes with $\phi_1 = 0.3, 0.5$, and 0.7. Because the RAR is independent of the variance of the innovations, a value such as unity may be used for σ_a^2 in the simulation study. In Tables 8–10 the expected values of the RAR and corresponding standard deviations in parentheses are given for the three Markov models. A comparison of the third column in

TABLE 9. Expected Value of the RAR for a Markov Model With $\phi_1 = 0.5$

N	$E(\bar{R}_N^*)$ Asymptotic	$E(\bar{R}_N^*)$ Simulated*
5	4.8541	2.0194 (0.0025)
10	6.8647	3.5438 (0.0061)
15	8.4075	4.8738 (0.0092)
20	9.7081	6.0432 (0.0120)
25	10.8540	7.1131 (0.0147)
30	11.8900	8.0779 (0.0171)
35	12.8426	8.9858 (0.0194)
40	13.7294	9.8655 (0.0212)
45	14.5622	10.6837 (0.0233)
50	15.3499	11.4170 (0.0252)
60	16.8150	12.8455 (0.0283)
70	18.1622	14.1721 (0.0320)
80	19.4163	15.4320 (0.0350)
90	20.5941	16.5726 (0.0376)
100	21.7080	17.5991 (0.0400)
125	24.2703	20.2124 (0.0459)
150	26.5868	22.5342 (0.0515)
175	28.7170	24.6356 (0.0562)
200	30.6998	26.6039 (0.0603)

*The parenthetical values are standard deviations.

these tables reveals that the expected value of the RAR increases for increasing N and ϕ_1.

It is also possible to compare the estimated expected value of the RAR for a Markov model to an analytical large-sample approximation that is given by *Siddiqui* [1976] as

$$E(\bar{R}_N^*) = \{(\pi N/2)[(1 - \phi_1^2)/(1 - \phi_1)^2]\}^{1/2} \quad (58)$$

In Tables 8–10 the output from (58) for the three Markov models are cataloged. A perusal of these tables demonstrates that Siddiqui's approximation for $E(\bar{R}_N^*)$ is not too accurate for the cases considered, and the precision decreases for increasing ϕ_1.

Conclusions

There is no doubt that the pursual of possible explanations to the Hurst phenomenon has stimulated valuable research both in hydrology and in mathematical statistics. In addition to the Hurst coefficient K defined in (17), other coefficients have been suggested to model the generalized Hurst coefficient h given in (15). For example, *Gomide* [1975], *Siddiqui* [1976], *Anis and Lloyd* [1976], and *Wallis and Matalas* [1970] proposed alternative procedures to model h. One of the major reasons for developing alternative exponents to K was to produce a coefficient that would reach its limiting value of 0.5 more quickly than K would. Nevertheless, it must be borne in mind that the definition of the Hurst phenomenon is based on a comparison of the value of K in small and moderate samples to its large sample value of 0.5. If the empirical, or theoretical, value of another estimate of h is compared for finite time series length to its asymptotic magnitude of $\frac{1}{2}$, the Hurst phenomenon should probably be redefined in terms of that statistic. However, because of the inherent statistical properties of the RAR the authors recommend that future research primarily be devoted to the study of this statistic and that less emphasis be put on the various definitions of the Hurst coefficient.

Feller [1951] proved that the asymptotic formula for the expected value of the adjusted range in (18) is valid for IID random variables. As is shown in (19) for large samples, Feller's equation is also correct for the expected value of the RAR for IID summands. The exact analytical expression for the expected value of the RAR for NID summands was derived by *Anis and Lloyd* [1976] and is written in (28). For finite samples the simulation and analytical results of Table 5 indicate that the expected values of the RAR and hence K are functions of the sample size but are virtually independent of the underlying distribution for IID summands. Accordingly, it has been suggested that the Hurst phenomenon could be explained by a combination of transience and autocorrelation [*Wallis and O'Connell,* 1973]. This implies that perhaps either a short-memory or a long-memory model that takes into account the autocorrelation structure of a time series may explain the Hurst phenomenon. Perhaps a better way to phrase this is that if a given stochastic model that is fit to a given data set preserves the important historical statistics such as the RAR and K, then that model may indirectly account for the Hurst phenomenon. Therefore it can be argued that a resolution to the controversies related to the Hurst phenomenon boils down to determining stochastic models that preserve the RAR, as well as other relevant historical statistics.

If a stochastic model is to retain the historical statistical characteristics of a time series, then the model must provide a good statistical fit to the data. This can be accomplished in practice by following the identification, estimation, and diagnostic check stages of model construction. For long-memory FGN processes the authors have suggested an efficient estimation procedure that uses the method of ML, and they also have developed a technique for calculating the model residuals so that they can be tested by appropriate diagnostic checks. In addition, (40) provides a means for exactly simulating FGN such that the synthetic traces will be outside the Brownian domain for the parameter H in the range $0.5 < H < 1$. The Fortran computer algorithms for these improved FGN procedures are listed by *Hipel and McLeod* [1978*b*] in part 3.

Short-memory models provide an alternative approach to FGN processes for modeling hydrological time series. In particular, the Box-Jenkins family of short-memory models possesses great potential for widespread applications to water resource and other geophysical problems. *Hipel et al.* [1977*a*] have provided some new procedures in Box-Jenkins modeling to simplify and also substantiate the three stages of model development. *McLeod et al.* [1977] have demonstrated the utility of the contemporary Box-Jenkins modeling techniques of *Hipel et al.* [1977*a*] by applying these procedures to both nonseasonal and seasonal time series. In addition, *McLeod and Hipel* [1978*b*] have developed improved simulation techniques for Box-Jenkins models and have provided the Fortran computer listings for the simulation methods.

TABLE 10. Expected Value of the RAR for a Markov Model With $\phi_1 = 0.7$

N	$E(\bar{R}_N^*)$ Asymptotic	$E(\bar{R}_N^*)$ Simulated*
5	6.6713	2.0435 (0.0025)
10	9.4346	3.7235 (0.0059)
15	11.5550	5.2915 (0.0091)
20	13.3425	6.7304 (0.0123)
25	14.9174	8.0874 (0.0154)
30	16.3412	9.3309 (0.0184)
35	17.6505	10.5117 (0.0212)
40	18.8692	11.6603 (0.0235)
45	20.0138	12.7462 (0.0262)
50	21.0964	13.7239 (0.0286)
60	23.1100	15.6339 (0.0331)
70	24.9616	17.4191 (0.0378)
80	26.6851	19.1225 (0.0419)
90	28.3038	20.6666 (0.0454)
100	29.8348	22.0685 (0.0490)
125	33.3564	25.6001 (0.0570)
150	36.5401	28.7578 (0.0648)
175	39.4678	31.6509 (0.0716)
200	42.1928	34.3412 (0.0772)

*The parenthetical values are standard deviations.

The question arises of whether a short-memory or a long-memory process should be selected to model a given time series. The AIC provides one means of model discrimination based on the principles of good statistical fit and parsimony of the model parameters. For the six annual river flow time series considered in this paper the results of Table 4 show that in all six cases the AIC chooses the best fitting Arma model in preference to the FGN process. Although there may be certain situations where the FGN model is appropriate to use, the inherent inflexibility of a FGN process may limit the use of this model in many types of practical applications. Rather than allowing for a choice of the required number of model parameters to use in a given situation as is done in Box-Jenkins modeling, the FGN model is always restricted to just three parameters (i.e., the mean, the variance, and H).

In part 2, *Hipel and McLeod* [1978*a*] demonstrate that Arma models do preserve the RAR or equivalently K. By fitting Arma models to 23 geophysical time series and by using Monte Carlo techniques it is shown that the observed RAR or K is retained by the models. Therefore in many practical situations it may be unnecessary to employ FGN models in order to preserve the RAR.

Notation

a coefficient used with the RAR in (15).
a_t white noise component in an Arma model
a_H coefficient used in (33).
b_H coefficient used in (34).
$B_H(t)$ fractional Brownian motion.
$\mathbf{C}_N(H)$ correlation matrix for a FGN process.
D_N' general deviation.
D_N crude deviation.
D_N^* standard deviation.
e_t white noise that is NID (0, 1).
$\mathbf{e}$ column vector containing the time series $e_1, e_2, \cdots, e_N$.
$E(z_t)$ expected value of z_t.
FGN discrete fractional Gaussian noise.
G Hurst estimate for H in a FGN model.
$GH(10)$ type of G Hurst estimate for H.
h generalized Hurst coefficient.
GH_i magnitude of $GH(10)$ for the ith simulated series of length N.
$h(N)$ local Hurst exponent of *Anis and Lloyd* [1976].
H parameter in a FGN model.
$\hat{H}_i$ MLE of H for the ith simulated series of length N.
IID identically independently distributed random variable.
K Hurst estimate of h.
K' Hurst coefficient calculated by using (54).
$\log L(\mu, \gamma_0, H)$ log likelihood of μ, γ_0, and H.
$\log L_{\max}(H)$ maximized log likelihood function of H.
m_{ij} typical element of $\mathbf{M}$.
m_N' general deficit.
m_N crude deficit.
m_N^* adjusted deficit.
$\mathbf{M}$ lower triangular matrix obtained from $\mathbf{C}_N(H)$ by Cholesky decomposition.
M_N' general surplus.
M_N crude surplus.
M_N^* adjusted surplus.
MLE maximum likelihood estimate.
MSE mean square error.
N number of data in the z_t series.
$\bar{N}$ number of sequences of length N.
NID normally independently distributed variable.
R_N' general range.
$\bar{R}_N'$ rescaled general range.
R_N crude range.
$\bar{R}_N$ rescaled crude range.
R_N^* adjusted range.
$\bar{R}_N^*$ rescaled adjusted range.
$\bar{R}_{r'}^*(t, r)$ RAR of the subseries $z_t, z_{t+1}, \cdots, z_r$, where $r' = r - t + 1$.
RE relative efficiency.
S_k' kth general partial sum.
S_k kth crude partial sum.
S_k^* kth adjusted partial sum.
$S(\mu, H)$ function defined in (42).
t discrete time, continuous time when dealing with $B_H(t)$.
T statistic time.
$\bar{T}$ mean of T.
T_i value of T for the ith simulated sequence.
V_T variance of T.
YH Gomide estimate of h.
YH' Gomide estimate of h obtained by using (28) and (56).
z_t time series value at time t.
$\bar{z}_N$ mean of the series $z_1, z_2 \cdots, z_N$.
2, $\cdots, z_N$.
$\mathbf{z}$ column vector containing $z_1, z_2, \cdots, z_N$.
α adjustment factor.
β measure of skewness in a stable distribution.
γ_0 population variance.
γ_k TACVF at lag k.
Γ gamma function.
μ population mean; location parameter for a stable distribution.
ρ_k TACF at lag k.
σ scale parameter for a stable distribution.
σ_a^2 variance of a_t.
ν characteristic exponent of a stable distribution.
ϕ_1 nonseasonal autoregressive parameter at lag 1.
$\omega(u)$ characteristic function of a stable distribution.
$\mathbf{1}$ column vector with N values of unity.

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