DIAGNOSTIC CHECKING ARMA TIME SERIES MODELS USING SQUARED-RESIDUAL AUTOCORRELATIONS

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Abstract. Squared-residual autocorrelations have been found useful in detecting non-linear types of statistical dependence in the residuals of fitted autoregressive-moving average (ARMA) models (Granger and Andersen, 1978; Miller, 1979). In this note it is shown that the normalized squared-residual autocorrelations are asymptotically unit multivariate normal. The results of a simulation experiment confirming the small-sample validity of the proposed tests is reported.

Keywords. ARMA time series; diagnostic checking; nonlinear time series; portmanteau test; testing for statistical independence.

1. INTRODUCTION AND SUMMARY

The ARMA (p, q) model for n observations $z_1, \ldots, z_n$ of a stationary mean μ time series can be written

$$\phi(B)(z_t - \mu) = \theta(B)a_t, \tag{1.1}$$

where

$$\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p,$$
$$\theta(B) = 1 - \theta_1 B - \cdots - \theta_q B^q,$$

where μ is the series mean and B is the backshift operator on t. The polynomials $\phi(B)$ and $\theta(B)$ are assumed to have all roots outside the unit circle and to have no factors in common. The standard large-sample estimation theory (Whittle, 1961; Hannan, 1970) requires that the a_t's be independent and identically distributed with finite variance.

A very useful procedure for checking the adequacy of a fitted ARMA model is based on testing the estimated innovations or residuals, $\hat{a}_t$, for whiteness. Box and Pierce (1970) obtained the distribution of the residual autocorrelations function

$$\hat{r}_a(k) = \sum_{k+1}^{n} \hat{a}_t \hat{a}_{t-k} \Big/ \sum_{1}^{n} \hat{a}_t^2 \tag{1.2}$$

0143-9782/83/04 0269-05 $02.50/0
JOURNAL OF TIME SERIES ANALYSIS Vol. 4, No. 4

and suggested the portmanteau statistic

$$Q_a = n \sum_{i=1}^{M} \hat{r}_a^2(i) \tag{1.3}$$

for testing the whiteness of the residuals. Under the assumption of model adequacy, Q_a is approximately $\chi^2(M-p-q)$ provided M and n are large enough (McLeod, 1978). Davies, Triggs and Newbold (1977) and Ljung and Box (1978) demonstrated that the modified statistic

$$Q_a^* = n(n+2) \sum_{i=1}^{M} \hat{r}_a^2(i)/(n-i) \tag{1.4}$$

provides a closer small-sample approximation to $\chi^2(M-p-q)$.

Granger and Andersen (1978) suggested that the autocorrelation function of the square of a time series could be useful in identifying non-linear bilinear time series. Granger and Andersen (1978, p. 86) found some series modelled in Box and Jenkins (1976) in which the squared residuals appear to be autocorrelated even though the residuals do not. In this situation, Granger and Andersen suggested that improved forecasts could be obtained by fitting a simple bilinear model to the residuals of the fitted ARMA model. Nonlinear time series modelling methods of Yakowitz (1979a, b) and Tong and Lim (1980) may also prove useful in this situation. Miller (1979) also reported, when modelling a mean daily riverflow series, that the residuals of a fitted ARMA did not appear to be autocorrelated although the squared residuals seemed significantly autocorrelated. When precipitation covariates were included in this model, Miller found that this difficulty was apparently eliminated. The present authors have also noticed numerous other hydrological and economic time series in which the squared residuals of the best fitting ARMA are significantly autocorrelated even though the usual residual autocorrelations do not suggest any model inadequacy.

The autocorrelation function of $\hat{a}_t^2$ is estimated by

$$\hat{r}_{aa}(k) = \sum_{t=k+1}^{n} (\hat{a}_t^2 - \hat{\sigma}^2)(\hat{a}_{t-k}^2 - \hat{\sigma}^2) \Big/ \sum_{t=1}^{n} (\hat{a}_t^2 - \hat{\sigma}^2)^2, \tag{1.5}$$

where

$$\hat{\sigma}^2 = \sum \hat{a}_t^2 / n.$$

In the next section, it is shown that for fixed M,

$$\sqrt{n}\, \hat{\mathbf{r}}_{aa} = (\hat{r}_{aa}(1), \ldots, \hat{r}_{aa}(M)) \tag{1.6}$$

is asymptotically normal as $n \to \infty$ with mean zero and unit covariance matrix. A significance test is provided by the portmanteau statistic

$$Q_{aa}^* = n(n+2) \sum_{i=1}^{M} \hat{r}_{aa}^2(i)/(n-i) \tag{1.7}$$

which is asymptotically $\chi^2(M)$ if the a_t are independent. In the final section, simulation experiments which suggest the small-sample applicability of these results are reported.

2. DISTRIBUTION OF SQUARED-RESIDUAL AUTOCORRELATIONS

Suppose that n observations, $z_1, \ldots, z_n$, of a time series are generated by an ARMA (p, q) model in which the innovations a_t are independent and identically distributed and for which $\langle a_t^8 \rangle$ exists, where $\langle \cdot \rangle$ denotes mathematical expectation. Let $\boldsymbol{\beta} = (\phi_1, \ldots, \phi_p, \theta_1, \ldots, \theta_q, \mu, \sigma^2)$ denote the true parameter values and let $\hat{\boldsymbol{\beta}}$ denote the least squares or Gaussian maximum likelihood estimates. Let $\dot{a}_t$, $t = 1, \ldots, n$ denote the residuals corresponding to arbitrary parameter values $(\dot{\phi}_1, \ldots, \dot{\phi}_p, \dot{\theta}_1, \ldots, \dot{\theta}_p, \dot{\mu})$ and let $\dot{\sigma}^2 = \sum \dot{a}_t^2 / n$.

The squared-residual autocorrelations for lag k can be written

$$\dot{r}_{aa}(k) = \dot{c}_{aa}(k) / \dot{c}_{aa}(0) \tag{2.1}$$

where

$$\dot{c}_{aa}(k) = \sum_{t=k+1}^{n} (\dot{a}_t^2 - \dot{\sigma}^2)(\dot{a}_{t-k}^2 - \dot{\sigma}^2)/n, \qquad k \geq 0. \tag{2.2}$$

Let $\hat{c}_{aa}(k)$ and $c_{aa}(k)$ be defined similarly.

The following lemma may be established by straightforward calculation (Li, 1981).

LEMMA

$$\partial \dot{c}_{aa}(k) / \partial \beta_i = \mathcal{O}_p(1/\sqrt{n}) \tag{2.3}$$

THEOREM. *For fixed M, $\sqrt{n}\, \hat{\mathbf{r}}_{aa}$ is asymptotically $N(\mathbf{0}, \mathbf{1}_M)$ as $n \to \infty$, where $\mathbf{1}_M$ is the M by M identity matrix.*

PROOF. Expanding $\dot{c}_{aa}(k)$ in a Taylor series about $\dot{\boldsymbol{\beta}} = \hat{\boldsymbol{\beta}}$,

$$\hat{c}_{aa}(k) = c_{aa}(k) + \sum_i (\hat{\beta}_i - \beta_i)\, \partial \dot{c}_{aa}(k) / \partial \beta_i + \mathcal{O}_p(1/n). \tag{2.4}$$

Since $\partial \dot{c}_{aa}(k)/\partial \beta_i = \mathcal{O}_p(1/\sqrt{n})$ and $\hat{\beta}_i - \beta_i = \mathcal{O}_p(1/\sqrt{n})$, it follows that

$$\hat{c}_{aa}(k) = c_{aa}(k) + \mathcal{O}_p(1/n). \tag{2.5}$$

From Theorem 14 of Hannan (1970, p. 228) $\sqrt{n}(c_{aa}(1), \ldots, c_{aa}(M)) / \gamma_{aa}(0)$ is asymptotically $N(\mathbf{0}, \mathbf{1}_M)$, where $\gamma_{aa}(0) = \langle (a_t^2 - \sigma^2)^2 \rangle$.

Expanding $\dot{r}_{aa}(k)$ about $\dot{c}_{aa}(0) = \gamma_{aa}(0)$ and $\dot{c}_{aa}(k) = \hat{c}_{aa}(k)$,

$$\hat{r}_{aa}(k) = \hat{c}_{aa}(k) / \gamma_{aa}(0) + \mathcal{O}_p(1/n). \tag{2.6}$$

The theorem now follows (using result 2c.4.12 of Rao 1973).

REMARK. The asymptotic variance of $\hat{r}_a(k)$ differs dramatically from that of $r_a(k)=\sum a_t a_{t-k}/\sum a_t^2$ due to the effect of estimating $\boldsymbol{\beta}$ (Box and Pierce, 1970; Dürbin, 1970; McLeod, 1978). However, it should be noted that this phenomenon does not occur with squared residuals.

3. SMALL-SAMPLE SIMULATION

The small-sample applicability of the results is examined for the AR (1) models:

$$z_t = \phi z_{t-1} + a_t, \tag{3.1}$$

where $t=1,\ldots,n$; $n=50$, 100, 200; $\phi=0$, $\pm.3$, $\pm.6$, $\pm.9$ and the a_t's are independent $N(0,1)$ random variables. The random number generator Superduper (Marsaglia, 1976), was used in conjunction with the transformation of Box and Muller (1958) to generate the a_t's. Each of the 21 models was simulated 10 000 times using an exact simulation technique (McLeod and Hipel, 1978). The parameter ϕ was estimated by the sample lag-one autocorrelation. The empirical variances of $\hat{r}_{aa}(1)$ and Q^*_{aa} with $M=20$ are shown in table I. Note that the variance of Q^*_{aa} is too large while that of $\hat{r}_{aa}(1)$ is too small. The approximation is much better for $n=200$ than $n=50$.

TABLE 1
EMPIRICAL BEHAVIOUR OF $\hat{r}_{aa}(1)$ AND Q^*_{aa}

		Number of Rejections at Nominal 5% Level		Empirical Mean	Empirical Variance	
n	ϕ	$\hat{r}_{aa}(1)$	Q^*_{aa}	Q^*_{aa}	$\hat{r}_{aa}(1)$	Q^*_{aa}
50	−.9	258	447	17.91	0.0164	49.00
50	−.6	298	458	18.03	0.0167	50.75
50	−.3	293	474	17.90	0.0167	50.71
50	0	255	520	17.98	0.0162	54.09
50	.3	278	448	17.83	0.0164	50.10
50	.6	305	490	17.97	0.0166	52.86
50	.9	290	456	17.84	0.0166	51.69
100	−.9	365	521	18.70	0.0087	49.82
100	−.6	352	509	18.64	0.0091	49.16
100	−.3	387	535	18.74	0.0092	49.01
100	0	372	492	18.65	0.0088	47.90
100	.3	373	492	18.65	0.0090	47.62
100	.6	349	473	18.57	0.0088	48.19
100	.9	375	516	18.55	0.0091	48.17
200	−.9	412	536	19.19	0.0046	46.06
200	−.6	399	497	19.03	0.0046	46.17
200	−.3	401	551	19.23	0.0046	47.49
200	0	423	502	19.23	0.0048	45.01
200	.3	450	535	19.26	0.0048	46.49
200	.6	428	494	19.18	0.0048	45.63
200	.9	418	492	19.18	0.0048	45.74

The empirical levels of tests using $\hat{r}_{aa}(1)$ and Q^*_{aa} at the nominal 5% level was also examined. Table I shows the number of times $|\hat{r}_{aa}(1)| > 1.96/\sqrt{n}$ and $Q^*_{aa} > 31.41$ ($M = 20$). The 95% confidence interval for the number of rejections is 500 ± 43 (Conover, 1971, p. 111). The test using Q^*_{aa} is slightly less than the lower limit 4 times whereas the test using $\hat{r}_{aa}(1)$ is always conservative although the approximation clearly improves with larger n.

The above experiments were repeated using the exact innovations to calculate $r_{aa}(1)$ and Q^*_{aa} instead of the residuals. As expected, no significant difference in the pattern of behaviour already described in Table I was found.

REFERENCES

BILLINGSLEY, P. (1961) The Lindeberg–Levy Theorem for Martingales. *Proceedings of the American Mathematical Society*, 12, 788–792.

BOX, G. E. P. and M. E. MULLER (1958) A Note on the Generation of Random Normal Deviates. *Ann. Math. Stat.* 29, 610–611.

BOX, G. E. P., M. E. MULLER and G. M. JENKINS (1976) *Time Series Analysis Forecasting and Control*, 2nd ed. Holden Day: San Francisco.

CONOVER, W. J. (1971) *Practical Nonparametric Statistics.* Wiley: New York.

DAVIES, N., C. M. TRIGGS and P. NEWBOLD (1977) Significance of the Box–Pierce Portmanteau Statistics in Finite Samples. *Biometrika*, 64, 517–522.

DURBIN, J. (1970) Testing for Serial Correlation in Least Squares Regression When Some of the Regressors are Lagged Dependent Variables. *Econometrika*, 38, 410–421.

GRANGER, C. W. and A. P. ANDERSEN (1978) *An Introduction to Bilinear Time Series Models.* Vandenhoeck and Ruprecht: Gottingen.

HANNAN, E. J. (1971) *Multiple Time Series.* Wiley: New York.

LI, W. K. (1981) *Topics in Time Series Modelling.* Ph.D. Thesis, The University of Western Ontario.

LJUNG, G. M. and G. E. P. BOX (1978) On a Measure of Lack of Fit in Time Series Models. *Biometrika*, 65, 297–303.

MARSAGLIA, G. (1976) Random Number Generation. In *Encyclopedia of Computer Science*, edited by A. Ralston. Petrocelli and Charter: New York, pp. 1192–1197.

MCLEOD, A. I. and K. W. HIPEL (1978) Simulation Procedures for Box–Jenkins Models. *Water Resources Research*, 14, 969–975.

MCLEOD, A. I. and K. W. HIPEL (1978) On the Distribution of Residual Autocorrelations in Box–Jenkins Models. *J. R. Statist. Soc. B* 296–302.

MILLER, R. B. (1979) Book review on 'An Introduction to Bilinear Time Series Models', by C. W. Granger and A. P. Andersen. *J. Amer. Statist. Ass.* 74, 927.

RAO, C. R. (1973) *Linear Statistical Inference and Its Applications*, 2nd ed. Wiley: New York.

TONG, H. and K. S. LIM (1980) Threshold Autoregression, Limit Cycles and Cyclical Data. *J. R. Statist. Soc. B* 42, 245–292.

WHITTLE, P. (1961) Gaussian Estimation in Stationary Time Series. *Bull. Int. Statist. Inst.* 33, 105–129.

YAKOWITZ, S. (1979a) A Nonparametric Markov Model for Daily River Flow. *Water Resources Research*, 15, 1035–1043.

YAKOWITZ, S. (1979b) Nonparametric Estimation of Markov Transition Functions. *Ann. Statist.* 7, 671–679.