Risk and Insurance Studies Centre
Student Research Competition
RISC SRC 2020

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Submissions should be sent by October 15, 2020, to src@riscinternational.org from a university e-mail address, be in one file and in pdf, be prepared in LaTeX, have a covering page with the name, university, year of study, and the university e-mail address of the submitter.

Fairly complete solutions – after presentations via video conference call – will receive RISC SRC 2020 Diplomas and, when appropriate, endorsements to apply to our graduate programs.

Undergraduate and Master's students are especially encouraged to participate.

Problem 1. Let $q : [0,1]^2 \to [0,1]$ be a non-decreasing in both arguments and Lipschitz continuous function, such that $q(0,x) = q(x,0) = 0$ and $q(x,1) = q(1,x) = x$ for all $x \in [0,1]$. Given any rectangle $\mathcal{R} := [x_1, x_2] \times [y_1, y_2] \subseteq [0,1]^2$, its q-volume is defined by

$$V_q(\mathcal{R}) = q(x_2, y_2) - q(x_2, y_1) - q(x_1, y_2) + q(x_1, y_1). \tag{1}$$

Prove that irrespective of q and $\mathcal{R}$, q-volume (1) is always in the interval $[-1/3, 1]$. Are the end-points of the interval $[-1/3, 1]$ attainable for some q and $\mathcal{R}$? Can there be a negative q-volume?

Problem 2. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a random variable $X : \Omega \to \mathbb{R}$, the Value-at-Risk of X at level $\alpha \in (0,1)$ is defined by $\mathrm{Var}_\alpha(X) = \inf\{m \in \mathbb{R} : \mathbb{P}(X + m < 0) \le \alpha\}$. If $X \in L^1$, then its Expected Shortfall at level $\alpha \in (0,1]$ is defined by $\mathrm{ES}_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha \mathrm{Var}_\beta(X)\mathrm{d}\beta$. Show that if random variables $X_n \in L^1$, $n \in \mathbb{N}$, and $X \in L^1$ are such that $X_n \xrightarrow{\text{a.s.}} X$ when $n \to \infty$, then

$$\mathrm{ES}_\alpha(X) \le \liminf_n \mathrm{ES}_\alpha(X_n) \tag{2}$$

whenever $a \in (0,1)$. Construct a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and random variables $X_n \in L^1$, $n \in \mathbb{N}$, and $X \in L^1$ satisfying $X_n \xrightarrow{\text{a.s.}} X$ such that statement (2) fails when $\alpha = 1$.

Problem 3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ be two random variables. Denote their probability laws by $\mathcal{L}_X$ and $\mathcal{L}_Y$, and let F and G denote the respective distribution functions. When holds, the integration by parts formula (equation) is

$$\int_{(a,b]} G(x)\mathcal{L}_X(\mathrm{d}x) = F(b)G(b) - F(a)G(a) - \int_{(a,b]} F(x)\mathcal{L}_Y(\mathrm{d}x). \tag{3}$$

Prove that equation (3) holds when F and G have no common points of discontinuity in the interval $(a,b] \subseteq \mathbb{R}$, and argue whether or not the condition of having no common points of discontinuity in $(a,b]$ is necessary for the validity of equation (3). Furthermore, argue whether or not the semi-closed integration interval $(a,b]$ can be replaced by the closed interval $[a,b]$ in equation (3).

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