Statistics 4657/9657: Assignment 1

Handout : September 5, 2008.
Due date: September 17, 2008

1. Suppose $\{a_n\}_{n\geq 1}$ is a sequence of real numbers which converges to a constant. In the examples below it will be sufficient to consider the limit to be 0.

   Give examples such sequences satisfying $a_n \to 0$ as $n \to \infty$ and the criteria below.

   (a) Give an example for which na_n tends to infinity.

   (b) Give an example for which $\{na_n\}$ does not converge (to any finite number or $\pm\infty$), and verify this for your example.

   Hint : A sequence fails to converge to any finite number or $\pm\infty$ if there are at least two limit points.

   (c) Give an example for which $na_n \to \infty$ but for which $\sqrt{n}a_n$ converges.

2. This problem deals with some applications of Taylor's Theorem with remainder.

   (a) State this Theorem, with reference.

   In the problems below state clearly how you are using this Theorem.

   (b) Suppose $f : \mathcal{R} \mapsto \mathcal{R}$, and that f has continuous derivative at x_0. For constant a, and the sequence $a_n = \frac{a}{\sqrt{n}}$ prove

$$\sqrt{n}\,(f(x_0 + a_n) - f(x_0)) \to f'(x_0)a \ .$$

   Show where you are using continuity of the first derivative in your proof.

   Remark : There is no assumption about derivatives of order higher than one, so you will have to use Taylor's Theorem with remainder.

   This example is related to 1, since $f(x_0 + a_n) - f(x_0) \to 0$, but when scaled by the factor $\sqrt{n}$ it does converge, but not converge to 0. Without the assumption of continuity of f' at x_0 the scaled sequence might not even converge. Can you give an example?

(c) Extend the above to the case where a_n has the property that $\sqrt{n}a_n \to a$ (rather than requiring that $\sqrt{n}a_n = a$ for all n).

Remark: Taylor's Theorem with remainder allows us to deduce the rate of convergence of $f(x_0+a_n)$ to $f(x_0)$ from the rate of convergence of a_n. This property is used in probability to study convergence of functions of estimators, amongst others, in results such as the delta method.

(d) Now suppose that f has two continuous derivatives at x_0. Suppose also that $f'(x_0) = 0$. Prove

$$n\left(f(x_0 + a_n) - f(x_0)\right) \to \frac{a^2}{2} f''(x_0) \ .$$

Remark : There is no assumption about derivatives of order higher, so you will have to use Taylor's Theorem with remainder.

3. Prove the following.

(a) Suppose $a_n \to a \in \mathcal{R}$.

$$b_n = \prod_{i=1}^{n} \left(1 + \frac{a_n}{n}\right) \to e^a \ .$$

Remark : You will use the idea and application of Taylor's Theorem with remainder as in the previous problem. It is easier to work with $\log(b_n)$. Suppose f is a function with two continuous derivatives at 0.

(b) Suppose $a_n \to 0$ and $\sqrt{n}a_n \to a \in \mathcal{R}$. Let f be a function with two continuous derivatives, $f(0) = 1$, $f'(0) = 0$ and $f''(0) = 1$. Find $\lim_{n\to\infty} f(a_n)^n$. Where are you using the continuity of f''?

4. Consider the function

$$f(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \ . \end{cases}$$

Let $(a_n)_{n\geq 1}$ be a sequence that converges to 0. Then $f(a_n) \to f(0) = 0$ as $n \to \infty$. If we know the rate at which a_n converges, does $f(a_n)$ converge at this same rate, as it would from the methods in question 2 and Taylor's Theorem with remainder? The answer is no.

(a) Verify that f is continuous at 0, and hence that $f(a_n) \to f(0) = 0$.

(b) Show f has derivative at all x except $x = 0$.

(c) Consider the sequence $a_n = \frac{(-1)^n}{n}$ which converges to 0 at rate n. Prove that the sequence $(nf(a_n))_{n\geq 1}$ does not converge.

(d) Consider sequences $(a_n)_{n\geq 1}$ which converge to 0, but for which $na_n \to a$.

If $a > 0$ show that $nf(a_n)$ converges.

If $a < 0$ show that $nf(a_n)$ converges.

Remark : These examples show that even if a_n converges to 0 at a known rate (n is these examples), then $nf(a_n)$ might converge or might not converge.

5. Two important notions that we use later in this course are Cauchy sequences and the Bolzano-Weierstrass property of sets of real numbers.

   (a) State these two definitions with reference and page number.

   (b) Prove that if a sequence $(a_n)_{n\geq 1}$ converges then it is a Cauchy sequence.

   (c) Give an example of a sequence which satisfies the Balzano-Weierstrass property but is not a convergent sequence.

   (d) Suppose that a sequence $(a_n)_{n\geq 1}$ has two limit points. Prove that it is not a Cauchy sequence.

6. There are several different notions of convergence. In this course we study several types related to probability and measure. In real analysis one type of convergence is convergence in Cesaro mean.

   Definition : A sequence $(a_n)_{n\geq 1}$ of real numbers converges in Cersaro mean if and only if the sequence $(b_n)_{n\geq 1}$ converges as a sequence of real numbers, where

$$b_n = \frac{1}{n}\sum_{i=1}^{n} a_i \ .$$

   (a) Suppose that $(a_n)_{n\geq 1}$ converges to a finite limit a. Prove that the sequence also converges in Cesaro mean and this limit is also a.

   (b) Consider $a_n = (-1)^n$. Prove this sequence does not converge, but that it converges in Cesaro mean.

7. Give the definition of lim inf and lim sup for sequences of real numbers, with reference.

   Prove that a sequence $(a_n)_{n\geq 1}$ of real numbers converges to a finite number a if and only if $\liminf_{m\to\infty} a_m = a$ and $\limsup_{m\to\infty} a_m = a$.