Statistics 3657a: Chapter 2.3 Discrete Example

Suppose $X \sim$ is a random variable with pmf $p : \mathcal{R} \mapsto [0, 1]$.

Consider a transformation $Y = g(X)$, where g is a function $g : \mathcal{D} \mapsto \mathcal{R}$. The domain $\mathcal{D}$ of g is typically the reals $\mathcal{R}$, but it must at least contain the range of X, that is the set of values that X can take on.

How can we find the distribution of Y?

What kind of a r.v. is Y? Is it discrete or continuous? It is discrete. To see this note the set of possible values $E = \{y : Y = y \text{ with } P(Y = y) > 0\}$ has the property that if $y \in E$ then there must exist at least one x

$$P(X = x) > 0$$

and $y = g(x)$ for this x. How many such numbers x are there that satisfies both these properties? There may be one, possibly several, but at most only countably many of these. The E contains at most as many different numbers as the possible values of x for which $P(X = x) > 0$. Thus E is either a set of finite size, or one that is countable, that is there is a 1 to 1 correspondence with the set of integers and hence we can write E in the form

$$E = \{y_1, y_2, y_3, \ldots\} .$$

Since Y is discrete we can describe its distribution using either a cdf or pmf. We will typically calculate the pmf of Y. For notation we write this pmf as p_Y, and will now need to describe p_Y in terms of what is given, namely the distribution or pmf of X, say p_X.

For any given y we can (in prinicple) write the event

$$\{Y = y\} = \{X \in \{x : g(x) = y\}\}$$

Thus by the third axiom of probability

$$\begin{aligned} p_Y(y) &= P(Y = y) \\ &= P(X \in \{x : g(x) = y\}) \\ &= \sum_{x:g(x)=y} P(X = x) \end{aligned}$$

$$= \sum_{x:g(x)=y} p_X(x)$$

The only difficulty in calculation is finding the set $A_y = \{x : g(x) = y\}$ for all possible y. For most real values y the set A_y is the empty set.

Example

Suppose X has the discrete uniform uniform distribution on the set $\{-2, -1, 0, 1, 2\}$. Consider the r.v. $y = X^2$. What is the distribution of Y?

Here we need to consider the function $g : x \mapsto x^2$. Thus we will need to consider first the possible values that Y can take with positive probability. Equivalently what values under the rule or function g does the range of $X = \text{Range}(X) = \{-2, -1, 0, 1, 2\}$ get mapped to? This set gets mapped to $\{0, 1, 4\}$.

The student should finish this problem at home.