University of Western Ontario

Statistics 3657 Term Test I

October 8, 2014, 5:30 PM - 6:20 PM

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Total (50)	

Name : ______________________________

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1. (a) Suppose A_1, A_2 and B are events and that $P(B) > 0$. Suppose also that A_1 and A_2 are disjoint. Using the axioms of probability and the definition of conditional probability prove that

$$P(A_1 \cup A_2 | B) = P(A_1|B) + P(A_2|B) \ .$$

(b) One of our Theorems from class is

Suppose X, Y are independent r.v.s and that g, h are functions (with appropriate domains). Then the r.v.s $g(X)$ and $h(Y)$ are independent r.v.s.

Prove this theorem in the discrete case, that is X, Y are discrete r.v.s.

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2. **This problem continues on the next page**

   (a) Suppose that X has a pdf

   $$f_X(x) = ce^{-x} I_{[0,1]}(x) \ .$$

   Find the constant c so that this is a pdf. Sketch the pdf f_X.

   Let $Z = e^{-X}$. Find the pdf of Z in a complete form, that is for all real arguments. In particular give the support as needed. Sketch the pdf of Z.

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$$1 = \int_{\mathbb{R}} f_X(x)\,dx = c\int_0^1 e^{-x}\,dx = c\cdot(-1)e^{-x}\Big|_0^1 = c\left(-e^{-1}+1\right)$$

$$\therefore \quad c = \frac{1}{1-e^{-1}}$$

$h(x) = e^{-x}$, $h'(x) = -e^{-x} < 0$, so h is a monotone decreasing function

$h^{-1}(z) = -\ln(z)$

$\dfrac{dh^{-1}(z)}{dz} = -\dfrac{1}{z}$

$$\therefore \quad f_Z(z) = f_X(\ln(z)) \cdot |h^{-1\prime}(z)|$$

$$= c \cdot e^{-(-\ln(z))} \cdot \left|\frac{1}{z}\right| \quad \text{for } 0 < -\ln(z) < 1 \iff \frac{1}{e} < z < 1$$

$$= c.$$

$$\therefore \quad f_Z(z) = \begin{cases} \dfrac{1}{1-e^{-1}} & \text{if } e^{-1} < z < 1 \\ 0 & \text{o/w} \end{cases}$$

Aside : $Z \sim \text{Unif}(e^{-1}, 1)$

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(b) Suppose X and Y have bivariate pdf f. Let $T = X + Y$. State the formula for the pdf of T in terms of the pdf's f the joint pdf of X, Y.

(c) Now suppose that X, Y are independent and each have pdf f_X given in the first part of this problem. Find the pdf of T. Sketch the pdf of T.

If you were not able to find the constant c in part (a), just write this constant as c and continue the problem and integration in that form.

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(c) $f_T(t) = \int_{-\infty}^{\infty} f(x) f(t-x)\, dx$

The limits of integration are obtained from the set $A_t = \{ x \mid f(x) f(t-x) > 0 \}$.

$$f(x) f(t-x) = c^2 e^{-x} e^{-(t-x)} I(0<x<1,\ 0<t-x<1)$$

$$= c^2 \cdot e^{-t}\, I(0<x<1,\ 0<t-x<1)$$

$$A_t = [0,1] \cap [t-1, t]$$

$$= \begin{cases} \phi & \text{if } t<0, \text{ or } t-1>1 \\ [0,t] & \text{if } 0 \le t \le 1 \\ [t-1,1] & \text{if } 1 < t \le 2 \end{cases}$$

The student can now finish this problem.

3. **This problem continues on the next page**

(a) Suppose that F is a continuous cdf (on the reals). Suppose that $X \sim F$. Then $F(X) \sim \text{Uniform}(0,1)$. Prove this result. If you choose to state a theorem from class to justify this then prove that theorem.

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(b) Suppose X is a r.v. with pmf

$$f(x) = \begin{cases} \frac{1}{6} & \text{if } x = 0 \\ \frac{2}{6} & \text{if } x = 1 \\ \frac{3}{6} & \text{if } x = 2 \\ 0 & \text{otherwise} \end{cases} .$$

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Find F = the CDF of X

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{6} & \text{if } 0 \le x < 1 \\ \frac{3}{6} & \text{if } 1 \le x < 2 \\ 1 & \text{if } x \ge 2 \end{cases}$$

(c) Give the distribution of $F(X)$.

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The r.v. $Y = F(X)$ is discrete, since X is discrete

Y can take values $F(0) = \frac{1}{6}$, $F(1) = \frac{1}{2}$, $F(2) = 1$ and no other values.

$\therefore$ the pmf of Y is

$$p_Y(y) = \begin{cases} \frac{1}{6} & \text{if } y = \frac{1}{6} \\ \frac{2}{6} & \text{if } y = \frac{1}{2} \\ \frac{3}{6} & \text{if } y = 1 \\ 0 & \text{o/w} \end{cases}$$

4. (a) Suppose X has pdf

$$f_X(x) = 3x^2 I(0 < x < 1) \,.$$

The conditional distribution of Y given that $X = x$ is Uniform(0 , x).

Based on this information determine if X and Y are independent, and justify your answer; no calculations are needed.

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Since $f_{Y|X=x}$ is not the same function for each $x \in \text{support}(f_X) = (0,1)$, therefore X and Y are dependent.

(b) Find the marginal pdf of Y.

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The joint pdf of X, Y is

$$f_{X,Y}(x,y) = \begin{cases} 3x^2 \cdot \frac{1}{x} & \text{if } 0 < x < 1 \text{ and } 0 < y < x \\ 0 & \text{o/w} \end{cases}$$

For $0 < y < 1$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

$$= \int_y^1 3x \, dx = \frac{3x^2}{2}\Big|_y^1 = \frac{3}{2}(1-y^3)$$

and $f_Y(y) = 0$ for $y \le 0$ or $y \ge 1$

$$\therefore f_Y(y) = \begin{cases} \frac{3}{2}(1-y^2) & \text{if } 0 < y < 1 \\ 0 & \text{o/w} \end{cases}$$