SS3858B Tutorial

Jan 20, 2010

1 Method of Moments Estimation of Hardy-Weinberg Example on Page 283.

Define

$$X = \begin{cases} 0 & \text{if} \quad M \\ 1 & \text{if} \quad MN \\ 2 & \text{if} \quad N \end{cases}$$

Then

$$\mu_1 = \mathrm{E}(X) = 0 \cdot (1-\theta)^2 + 1 \cdot 2\theta(1-\theta) + 2 \cdot \theta^2 = 2\theta.$$

The sample expectation

$$\hat{\mu}_1 = \frac{0 \times 342 + 1 \times 500 + 2 \times 187}{1029} = \frac{874}{1029}.$$

$\hat{\theta}$ is the solution of $\mu_1 = \hat{\mu}_1$, i.e.,

$$\hat{\theta} = \frac{874}{1029 \times 2} = 0.4246842.$$

R example for $n = 10000$ and $\theta = 0.6$

```
> sim=function(n,theta){
+     p=c((1-theta)^2,2*theta*(1-theta),theta^2)
+     u=runif(n)
+     r1=sum(u<=p[1])
+     r2=sum(u>p[1]&u<=p[1]+p[2])
+     r3=sum(u>p[1]+p[2])
+     c(r1,r2,r3)
+ }
> r=sim(10000,.6)
> r
[1] 1598 4693 3709
> sum(c(0,1,2)*r)/10000/2 #theta.hat
[1] 0.60555
```

2 Some Extra Examples on the Method of Moments Estimation

1. t distribution with d.f. $\nu > 2$. The mean is 0 and the variance is $\nu/(\nu - 2)$. Then
$$\mu_2 = \mathrm{E}(X^2) = \frac{\nu}{\nu - 2}.$$
$\hat{\nu}$ is the solution of
$$\frac{\nu}{\nu - 2} = \hat{\mu}_2,$$
where
$$\hat{\mu}_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2.$$
Therefore,
$$\hat{\nu} = \frac{2\hat{\mu}_2}{\hat{\mu}_2 - 1}.$$

Example for $t(5)$

```
> s=rt(10000,5)
> v=mean(s^2)
> 2*v/(v-1)
[1] 4.980385
```

2. Let $X_1, \cdots, X_n$ be iid with pdf
$$f(x|\theta) = \frac{1}{\theta}, \quad 0 \le x \le \theta, \quad \theta > 0.$$
Estimate θ using the method of moments.
$$\mathrm{E}(X) = \int_0^\theta x\frac{1}{\theta}dx = \frac{\theta}{2}.$$
Thus,
$$\hat{\theta} = 2\bar{X}.$$

3. Let $X_1, \cdots, X_n$ be iid Binomial(m, p), that is,
$$\mathrm{P}(X = x) = \binom{m}{x} p^x (1-p)^{m-x}, \quad x = 0, 1, \cdots, m.$$

Equating the first two sample moments to those of the population yields the system of equations

$$\begin{aligned}\bar{X} &= mp, \\ \frac{1}{n}\sum_{i=1}^{n} X_i^2 &= \underbrace{mp(1-p)}_{\text{Var(X)}} + \underbrace{m^2p^2}_{(\text{E}(X))^2}.\end{aligned}$$

We now obtain the method of moments estimators

$$\begin{aligned}\hat{p} &= 1 + \bar{X} - \frac{\frac{1}{n}\sum_{i=1}^{n} X_i^2}{\bar{X}}, \\ \hat{m} &= \frac{\bar{X}^2}{\bar{X} + \bar{X}^2 - \frac{1}{n}\sum_{i=1}^{n} X_i^2}.\end{aligned}$$

Numeric example for binomial$(10, 0.4)$

```
> s=rbinom(10000,10,.4)
> v=mean(s^2)
> x.bar=mean(s)
> 1+x.bar-v/x.bar #p.hat
[1] 0.4030938
> x.bar^2/(x.bar+x.bar^2-v) #m.hat
[1] 9.8843
```

3 Introduction to Delta Method

By Theorem B on Page 277,

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{\mathcal{D}} N(0, I(\theta)^{-1}).$$

Alternatively,

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{\mathcal{D}} N(0, n\text{Var}(\hat{\theta})).$$

Suppose g is a function where $g'(\theta)$ exists and $g'(\theta) \neq 0$. Applying first-order Taylor expansion and dropping higher order terms give the approximation

$$g(\hat{\theta}) \approx g(\theta) + g'(\theta)(\hat{\theta} - \theta).$$

That is,

$$\sqrt{n}(g(\hat{\theta}) - g(\theta)) \approx \sqrt{n}g'(\theta)(\hat{\theta} - \theta).$$

Thus,

$$\sqrt{n}(g(\hat{\theta}) - g(\theta)) \xrightarrow{\mathcal{D}} N\left(0, \frac{(g'(\theta))^2}{I(\theta)}\right),$$

or

$$\sqrt{n}(g(\hat{\theta}) - g(\theta)) \xrightarrow{\mathcal{D}} N(0, n(g'(\theta))^2 \mathrm{Var}(\hat{\theta})).$$

As an example, let us consider the distribution of $1/\bar{X}$, where $X_1, \cdots, X_n$ are iid from the population Normal(μ, σ^2) with σ^2 known. In this case,

$$\theta = \mu, \quad \hat{\theta} = \bar{X} \quad \text{and} \quad g(x) = \frac{1}{x}.$$

Then

$$\mathrm{Var}(\hat{\theta}) = \frac{\sigma^2}{n} \quad \text{and} \quad g'(x) = -\frac{1}{x^2}.$$

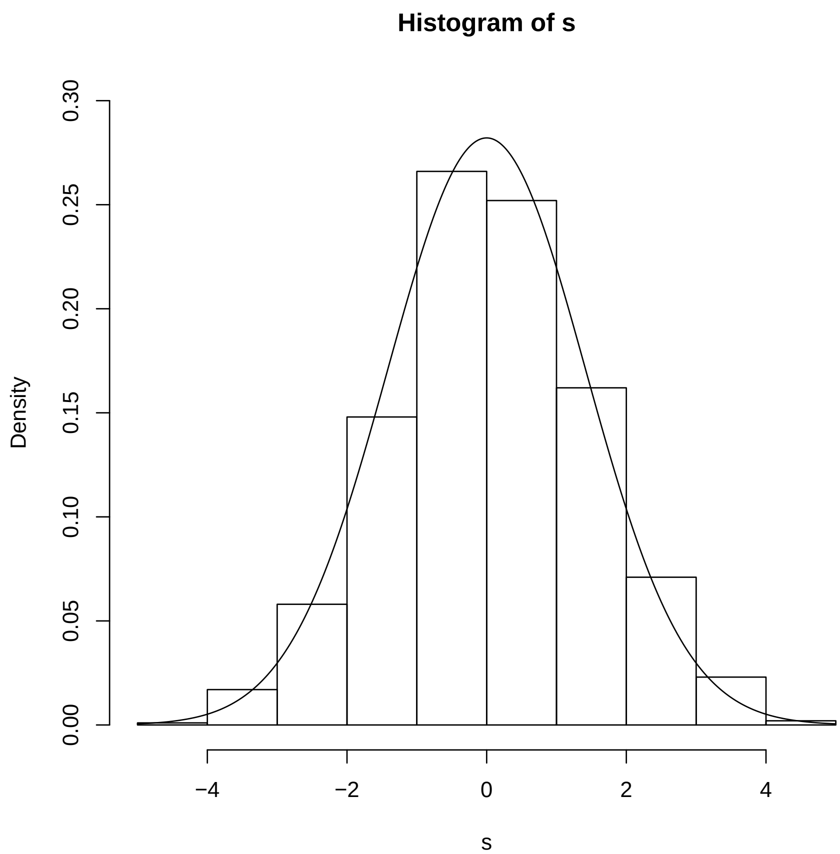
Therefore,

$$\sqrt{n}\left(\frac{1}{\bar{X}} - \frac{1}{\mu}\right) \xrightarrow{\mathcal{D}} N\left(0, \frac{\sigma^2}{\mu^4}\right).$$

Numeric example for $\mu = 1$, $\sigma^2 = 2$ and $n = 10000$ with 1000 replications

$$100\left(\frac{1}{\bar{X}} - 1\right) \xrightarrow{\mathcal{D}} N(0, 2).$$

```
> r=matrix(rnorm(10000*1000,1,sqrt(2)),nrow=1000)
> m=apply(r,1,mean)
> s=100*(1/m-1)
> mean(s)
[1] 0.05819064
> var(s)
[1] 2.095901
> hist(s,prob=T,ylim=c(0,0.3))
> x.co=seq(-5,5,len=1000) #x coordinates
> lines(x.co,dnorm(x.co,0,sqrt(2))) #theoretical density of N(0,2)
```


Histogram of s
Density
0.00
0.05
0.10
0.15
0.20
0.25
0.30
−4
−2
0
2
4
s