we now use the recursive identity (4.37) to determine $V_2(j)$ for each j; then $V_3(j)$ for each j; and so on, up to $V_n(j)$ for each j.

To obtain the maximizing sequence of states, we work in the reverse direction. Let j_n be the value (or any of the values if there are more than one) of j that maximizes $V_n(j)$. Thus j_n is the final state of a maximizing state sequence. Also, for $k < n$, let $i_k(j)$ be a value of i that maximizes $P_{i,j} V_k(i)$. Then

$$
\begin{aligned}
\max_{i_1,\ldots,i_n} P\{\mathbf{X}_n = (i_1,\ldots,i_n), \mathbf{S}^n = \mathbf{s}_n\}
&= \max_j V_n(j) \\
&= V_n(j_n) \\
&= \max_{i_1,\ldots,i_{n-1}} P\{\mathbf{X}_n = (i_1,\ldots,i_{n-1}, j_n), \mathbf{S}^n = \mathbf{s}_n\} \\
&= p(s_n|j_n) \max_i P_{i,j_n} V_{n-1}(i) \\
&= p(s_n|j_n) P_{i_{n-1}(j_n),j_n} V_{n-1}(i_{n-1}(j_n))
\end{aligned}
$$

Thus, $i_{n-1}(j_n)$ is the next to last state of the maximizing sequence. Continuing in this manner, the second from the last state of the maximizing sequence is $i_{n-2}(i_{n-1}(j_n))$, and so on.

The preceding approach to finding the most likely sequence of states given a prescribed sequence of signals is known as the *Viterbi Algorithm*.

Exercises

*1. Three white and three black balls are distributed in two urns in such a way that each contains three balls. We say that the system is in state $i, i = 0, 1, 2, 3$, if the first urn contains i white balls. At each step, we draw one ball from each urn and place the ball drawn from the first urn into the second, and conversely with the ball from the second urn. Let X_n denote the state of the system after the nth step. Explain why $\{X_n, n = 0, 1, 2, \ldots\}$ is a Markov chain and calculate its transition probability matrix.

2. Each individual in a population independently has a random number of offspring that is Poisson distributed with mean λ. Those initially present constitute the zeroth generation. Offspring of zeroth generation people constitute the first generation; their offspring constitute the second generation, and so on. If X_n denotes the size of generation n, is $\{X_n, n \geq 0\}$ a Markov chain. If it is, give its transition probabilities $P_{i,j}$; if it is not, explain why it is not.

3. There are k players, with player i having value $v_i > 0, i = 1, \ldots, k$. In every period, two of the players play a game, while the other $k-2$ wait in an ordered line. The loser of a game joins the end of the line, and the winner then plays a new game against the player who is first in line. Whenever i and j play, i wins with probability $\frac{v_i}{v_i+v_j}$.

(a) Define a Markov chain that is useful in analyzing this model.
(b) How many states does the Markov chain have?
(c) Give the transition probabilities of the chain.

4. Let **P** and **Q** be transition probability matrices on states $1, \ldots, m$, with respective transition probabilities $P_{i,j}$ and $Q_{i,j}$. Consider processes $\{X_n, n \geq 0\}$ and $\{Y_n, n \geq 0\}$ defined as follows:

(a) $X_0 = 1$. A coin that comes up heads with probability p is then flipped. If the coin lands heads, the subsequent states $X_1, X_2, \ldots$, are obtained by using the transition probability matrix **P**; if it lands tails, the subsequent states $X_1, X_2, \ldots$, are obtained by using the transition probability matrix **Q**. (In other words, if the coin lands heads (tails) then the sequence of states is a Markov chain with transition probability matrix **P** (**Q**).) Is $\{X_n, n \geq 0\}$ a Markov chain. If it is, give its transition probabilities. If it is not, tell why not.

(b) $Y_0 = 1$. If the current state is i, then the next state is determined by first flipping a coin that comes up heads with probability p. If the coin lands heads then the next state is j with probability $P_{i,j}$; if it lands tails then the next state is j with probability $Q_{i,j}$. Is $\{Y_n, n \geq 0\}$ a Markov chain. If it is, give its transition probabilities. If it is not, tell why not.

5. A Markov chain $\{X_n, n \geqslant 0\}$ with states $0, 1, 2$, has the transition probability matrix

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

If $P\{X_0 = 0\} = P\{X_0 = 1\} = \frac{1}{4}$, find $E[X_3]$.

6. Let the transition probability matrix of a two-state Markov chain be given, as in Example 4.2, by

$$\mathbf{P} = \begin{Vmatrix} p & 1-p \\ 1-p & p \end{Vmatrix}$$

Show by mathematical induction that

$$\mathbf{P}^{(n)} = \begin{Vmatrix} \frac{1}{2} + \frac{1}{2}(2p-1)^n & \frac{1}{2} - \frac{1}{2}(2p-1)^n \\ \frac{1}{2} - \frac{1}{2}(2p-1)^n & \frac{1}{2} + \frac{1}{2}(2p-1)^n \end{Vmatrix}$$

7. In Example 4.4 suppose that it has rained neither yesterday nor the day before yesterday. What is the probability that it will rain tomorrow?

8. An urn initially contains 2 balls, one of which is red and the other blue. At each stage a ball is randomly selected. If the selected ball is red, then it is replaced with a red ball with probability .7 or with a blue ball with probability .3; if the selected ball is blue, then it is equally likely to be replaced by either a red or blue ball.

(a) Let X_n equal 1 if the nth ball selected is red, and let it equal 0 otherwise. Is $\{X_n, n \geq 1\}$ a Markov chain? If so, give its transition probability matrix.

(b) Let Y_n denote the number of red balls in the urn immediately before the nth ball is selected. Is $\{Y_n, n \geq 1\}$ a Markov chain? If so, give its transition probability matrix.

(c) Find the probability that the second ball selected is red.

(d) Find the probability that the fourth ball selected is red.

***9.** In a sequence of independent flips of a coin that comes up heads with probability .6, what is the probability that there is a run of three consecutive heads within the first 10 flips?

10. In Example 4.3, Gary is currently in a cheerful mood. What is the probability that he is not in a glum mood on any of the following three days?

11. In Example 4.13, give the transition probabilities of the Y_n Markov chain in terms of the transition probabilities $P_{i,j}$ of the X_n chain.

12. For a Markov chain $\{X_n, n \geqslant 0\}$ with transition probabilities $P_{i,j}$, consider the conditional probability that $X_n = m$ given that the chain started at time 0 in state i and has not yet entered state r by time n, where r is a specified state not equal to either i or m. We are interested in whether this conditional probability is equal to the n stage transition probability of a Markov chain whose state space does not include state r and whose transition probabilities are

$$Q_{i,j} = \frac{P_{i,j}}{1 - P_{i,r}}, \qquad i, j \neq r$$

Either prove the equality

$$P\{X_n = m | X_0 = i, X_k \neq r, k = 1, \ldots, n\} = Q^n_{i,m}$$

or construct a counterexample.

13. Let $\mathbf{P}$ be the transition probability matrix of a Markov chain. Argue that if for some positive integer r, $\mathbf{P}^r$ has all positive entries, then so does $\mathbf{P}^n$, for all integers $n \geqslant r$.

14. Specify the classes of the following Markov chains, and determine whether they are transient or recurrent:

$$\mathbf{P}_1 = \begin{Vmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{Vmatrix}, \qquad \mathbf{P}_2 = \begin{Vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{Vmatrix},$$

$$\mathbf{P}_3 = \left\| \begin{matrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{matrix} \right\|, \qquad \mathbf{P}_4 = \left\| \begin{matrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 1 & 0 & 0 & 0 & 0 \end{matrix} \right\|$$

15. Prove that if the number of states in a Markov chain is M, and if state j can be reached from state i, then it can be reached in M steps or less.

***16.** Show that if state i is recurrent and state i does not communicate with state j, then $P_{ij} = 0$. This implies that once a process enters a recurrent class of states it can never leave that class. For this reason, a recurrent class is often referred to as a *closed* class.

17. For the random walk of Example 4.19 use the strong law of large numbers to give another proof that the Markov chain is transient when $p \neq \frac{1}{2}$.

Hint: Note that the state at time n can be written as $\sum_{i=1}^{n} Y_i$ where the Y_is are independent and $P\{Y_i = 1\} = p = 1 - P\{Y_i = -1\}$. Argue that if $p > \frac{1}{2}$, then, by the strong law of large numbers, $\sum_1^n Y_i \to \infty$ as $n \to \infty$ and hence the initial state 0 can be visited only finitely often, and hence must be transient. A similar argument holds when $p < \frac{1}{2}$.

18. Coin 1 comes up heads with probability 0.6 and coin 2 with probability 0.5. A coin is continually flipped until it comes up tails, at which time that coin is put aside and we start flipping the other one.

(a) What proportion of flips use coin 1?

(b) If we start the process with coin 1 what is the probability that coin 2 is used on the fifth flip?

(c) What proportion of flips land heads?

19. For Example 4.4, calculate the proportion of days that it rains.

20. A transition probability matrix $\mathbf{P}$ is said to be doubly stochastic if the sum over each column equals one; that is,

$$\sum_i P_{ij} = 1, \quad \text{for all } j$$

If such a chain is irreducible and consists of $M + 1$ states $0, 1, \ldots, M$, show that the long-run proportions are given by

$$\pi_j = \frac{1}{M+1}, \quad j = 0, 1, \ldots, M$$

***21.** A DNA nucleotide has any of four values. A standard model for a mutational change of the nucleotide at a specific location is a Markov chain model that supposes that in going from period to period the nucleotide does not change with probability $1 - 3\alpha$, and if it does change then it is equally likely to change to any of the other three values, for some $0 < \alpha < \frac{1}{3}$.

(a) Show that $P^n_{1,1} = \frac{1}{4} + \frac{3}{4}(1 - 4\alpha)^n$.
(b) What is the long-run proportion of time the chain is in each state?

22. Let Y_n be the sum of n independent rolls of a fair die. Find

$$\lim_{n \to \infty} P\{Y_n \text{ is a multiple of } 13\}$$

Hint: Define an appropriate Markov chain and apply the results of Exercise 20.

23. In a good weather year the number of storms is Poisson distributed with mean 1; in a bad year it is Poisson distributed with mean 3. Suppose that any year's weather conditions depends on past years only through the previous year's condition. Suppose that a good year is equally likely to be followed by either a good or a bad year, and that a bad year is twice as likely to be followed by a bad year as by a good year. Suppose that last year—call it year 0—was a good year.
(a) Find the expected total number of storms in the next two years (that is, in years 1 and 2).
(b) Find the probability there are no storms in year 3.
(c) Find the long-run average number of storms per year.
(d) Find the proportion of years that have no storms.

24. Consider three urns, one colored red, one white, and one blue. The red urn contains 1 red and 4 blue balls; the white urn contains 3 white balls, 2 red balls, and 2 blue balls; the blue urn contains 4 white balls, 3 red balls, and 2 blue balls. At the initial stage, a ball is randomly selected from the red urn and then returned to that urn. At every subsequent stage, a ball is randomly selected from the urn whose color is the same as that of the ball previously selected and is then returned to that urn. In the long run, what proportion of the selected balls are red? What proportion are white? What proportion are blue?

25. Each morning an individual leaves his house and goes for a run. He is equally likely to leave either from his front or back door. Upon leaving the house, he chooses a pair of running shoes (or goes running barefoot if there are no shoes at the door from which he departed). On his return he is equally likely to enter, and leave his running shoes, either by the front or back door. If he owns a total of k pairs of running shoes, what proportion of the time does he run barefooted?

26. Consider the following approach to shuffling a deck of n cards. Starting with any initial ordering of the cards, one of the numbers $1, 2, \ldots, n$ is randomly chosen in such a manner that each one is equally likely to be selected. If number i is chosen, then we take the card that is in position i and put it on top of the deck—that is, we put that card in position 1. We then repeatedly perform the same operation. Show that, in the limit, the deck is perfectly shuffled in the sense that the resultant ordering is equally likely to be any of the $n!$ possible orderings.

*27. Each individual in a population of size N is, in each period, either active or inactive. If an individual is active in a period then, independent of all else, that individual will be active in the next period with probability α. Similarly, if an individual is inactive in a period then, independent of all else, that individual will be inactive in the next period with probability β. Let X_n denote the number of individuals that are active in period n.

(a) Argue that $X_n, n \geqslant 0$ is a Markov chain.

(b) Find $E[X_n|X_0 = i]$.

(c) Derive an expression for its transition probabilities.

(d) Find the long-run proportion of time that exactly j people are active.

Hint for (d): Consider first the case where $N = 1$.

28. Every time that the team wins a game, it wins its next game with probability 0.8; every time it loses a game, it wins its next game with probability 0.3. If the team wins a game, then it has dinner together with probability 0.7, whereas if the team loses then it has dinner together with probability 0.2. What proportion of games result in a team dinner?

29. An organization has N employees where N is a large number. Each employee has one of three possible job classifications and changes classifications (independently) according to a Markov chain with transition probabilities

$$\begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.4 & 0.5 \end{bmatrix}$$

What percentage of employees are in each classification?

30. Three out of every four trucks on the road are followed by a car, while only one out of every five cars is followed by a truck. What fraction of vehicles on the road are trucks?

31. A certain town never has two sunny days in a row. Each day is classified as being either sunny, cloudy (but dry), or rainy. If it is sunny one day, then it is equally likely to be either cloudy or rainy the next day. If it is rainy or cloudy one day, then there is one chance in two that it will be the same the next day, and if it changes then it is equally likely to be either of the other two possibilities. In the long run, what proportion of days are sunny? What proportion are cloudy?

***32.** Each of two switches is either on or off during a day. On day n, each switch will independently be on with probability

$$[1 + \text{number of on switches during day } n - 1]/4$$

For instance, if both switches are on during day $n - 1$, then each will independently be on during day n with probability $3/4$. What fraction of days are both switches on? What fraction are both off?

33. Two players are playing a sequence of points, which begin when one of the players serves. Suppose that player 1 wins each point she serves with probability p, and wins each point her opponent serves with probability q. Suppose the winner of a point becomes the server of the next point.

(a) Find the proportion of points that are won by player 1.

(b) Find the proportion of time that player 1 is the server.

34. A flea moves around the vertices of a triangle in the following manner: Whenever it is at vertex i it moves to its clockwise neighbor vertex with probability p_i and to the counterclockwise neighbor with probability $q_i = 1 - p_i, i = 1, 2, 3$.
   (a) Find the proportion of time that the flea is at each of the vertices.
   (b) How often does the flea make a counterclockwise move that is then followed by five consecutive clockwise moves?

35. Consider a Markov chain with states 0, 1, 2, 3, 4. Suppose $P_{0,4} = 1$; and suppose that when the chain is in state $i, i > 0$, the next state is equally likely to be any of the states $0, 1, \ldots, i - 1$. Find the limiting probabilities of this Markov chain.

36. The state of a process changes daily according to a two-state Markov chain. If the process is in state i during one day, then it is in state j the following day with probability $P_{i,j}$, where

$$P_{0,0} = 0.4, \quad P_{0,1} = 0.6, \quad P_{1,0} = 0.2, \quad P_{1,1} = 0.8$$

Every day a message is sent. If the state of the Markov chain that day is i then the message sent is "good" with probability p_i and is "bad" with probability $q_i = 1 - p_i, i = 0, 1$
   (a) If the process is in state 0 on Monday, what is the probability that a good message is sent on Tuesday?
   (b) If the process is in state 0 on Monday, what is the probability that a good message is sent on Friday?
   (c) In the long run, what proportion of messages are good?
   (d) Let Y_n equal 1 if a good message is sent on day n and let it equal 2 otherwise. Is $\{Y_n, n \geqslant 1\}$ a Markov chain? If so, give its transition probability matrix. If not, briefly explain why not.

37. Show that the stationary probabilities for the Markov chain having transition probabilities $P_{i,j}$ are also the stationary probabilities for the Markov chain whose transition probabilities $Q_{i,j}$ are given by

$$Q_{i,j} = P_{i,j}^k$$

for any specified positive integer k.

38. Capa plays either one or two chess games every day, with the number of games that she plays on successive days being a Markov chain with transition probabilities

$$P_{1,1} = .2, \quad P_{1,2} = .8 \qquad P_{2,1} = .4, \quad P_{2,2} = .6$$

Capa wins each game with probability p. Suppose she plays two games on Monday.
   (a) What is the probability that she wins all the games she plays on Tuesday?
   (b) What is the expected number of games that she plays on Wednesday?
   (c) In the long run, on what proportion of days does Capa win all her games.

39. Consider the one-dimensional symmetric random walk of Example 4.19, which was shown in that example to be recurrent. Let π_i denote the long-run proportion of time that the chain is in state i.

(a) Argue that $\pi_i = \pi_0$ for all i.

(b) Show that $\sum_i \pi_i \neq 1$.

(c) Conclude that this Markov chain is null recurrent, and thus all $\pi_i = 0$.

40. A particle moves on 12 points situated on a circle. At each step it is equally likely to move one step in the clockwise or in the counterclockwise direction. Find the mean number of steps for the particle to return to its starting position.

***41.** Consider a Markov chain with states equal to the nonnegative integers, and suppose its transition probabilities satisfy $P_{i,j} = 0$ if $j \le i$. Assume $X_0 = 0$, and let e_j be the probability that the Markov chain is ever in state j. (Note that $e_0 = 1$ because $X_0 = 0$.) Argue that for $j > 0$

$$e_j = \sum_{i=0}^{j-1} e_i P_{i,j}$$

If $P_{i,i+k} = 1/3, k = 1, 2, 3$, find e_i for $i = 1, \ldots, 10$.

42. Let A be a set of states, and let A^c be the remaining states.

(a) What is the interpretation of

$$\sum_{i \in A} \sum_{j \in A^c} \pi_i P_{ij}?$$

(b) What is the interpretation of

$$\sum_{i \in A^c} \sum_{j \in A} \pi_i P_{ij}?$$

(c) Explain the identity

$$\sum_{i \in A} \sum_{j \in A^c} \pi_i P_{ij} = \sum_{i \in A^c} \sum_{j \in A} \pi_i P_{ij}$$

43. Each day, one of n possible elements is requested, the ith one with probability $P_i, i \geqslant 1, \sum_1^n P_i = 1$. These elements are at all times arranged in an ordered list that is revised as follows: The element selected is moved to the front of the list with the relative positions of all the other elements remaining unchanged. Define the state at any time to be the list ordering at that time and note that there are $n!$ possible states.

(a) Argue that the preceding is a Markov chain.

(b) For any state $i_1, \ldots, i_n$ (which is a permutation of $1, 2, \ldots, n$), let $\pi(i_1, \ldots, i_n)$ denote the limiting probability. In order for the state to be $i_1, \ldots, i_n$, it is necessary for the last request to be for i_1, the last non-i_1 request for i_2, the last non-i_1 or i_2 request for i_3, and so on. Hence, it appears intuitive that

$$\pi(i_1, \ldots, i_n) = P_{i_1} \frac{P_{i_2}}{1 - P_{i_1}} \frac{P_{i_3}}{1 - P_{i_1} - P_{i_2}} \cdots \frac{P_{i_{n-1}}}{1 - P_{i_1} - \cdots - P_{i_{n-2}}}$$

Verify when $n = 3$ that the preceding are indeed the limiting probabilities.

44. Suppose that a population consists of a fixed number, say, m, of genes in any generation. Each gene is one of two possible genetic types. If exactly i (of the m) genes of any generation are of type 1, then the next generation will have j type 1 (and $m - j$ type 2) genes with probability

$$\binom{m}{j} \left(\frac{i}{m}\right)^j \left(\frac{m-i}{m}\right)^{m-j}, \quad j = 0, 1, \ldots, m$$

Let X_n denote the number of type 1 genes in the nth generation, and assume that $X_0 = i$.

(a) Find $E[X_n]$.

(b) What is the probability that eventually all the genes will be type 1?

45. Consider an irreducible finite Markov chain with states $0, 1, \ldots, N$.

(a) Starting in state i, what is the probability the process will ever visit state j? Explain!

(b) Let $x_i = P\{\text{visit state } N \text{ before state } 0 | \text{start in } i\}$. Compute a set of linear equations that the x_i satisfy, $i = 0, 1, \ldots, N$.

(c) If $\sum_j j P_{ij} = i$ for $i = 1, \ldots, N - 1$, show that $x_i = i/N$ is a solution to the equations in part (b).

46. An individual possesses r umbrellas that he employs in going from his home to office, and vice versa. If he is at home (the office) at the beginning (end) of a day and it is raining, then he will take an umbrella with him to the office (home), provided there is one to be taken. If it is not raining, then he never takes an umbrella. Assume that, independent of the past, it rains at the beginning (end) of a day with probability p.

(a) Define a Markov chain with $r + 1$ states, which will help us to determine the proportion of time that our man gets wet. (*Note*: He gets wet if it is raining, and all umbrellas are at his other location.)

(b) Show that the limiting probabilities are given by

$$\pi_i = \begin{cases} \dfrac{q}{r+q}, & \text{if } i = 0 \\[2ex] \dfrac{1}{r+q}, & \text{if } i = 1, \ldots, r \end{cases} \qquad \text{where } q = 1 - p$$

(c) What fraction of time does our man get wet?

(d) When $r = 3$, what value of p maximizes the fraction of time he gets wet

*47. Let $\{X_n, n \geqslant 0\}$ denote an ergodic Markov chain with limiting probabilities π_i. Define the process $\{Y_n, n \geqslant 1\}$ by $Y_n = (X_{n-1}, X_n)$. That is, Y_n keeps track of the last two states of the original chain. Is $\{Y_n, n \geqslant 1\}$ a Markov chain? If so, determine its transition probabilities and find

$$\lim_{n\to\infty} P\{Y_n = (i, j)\}$$

48. Consider a Markov chain in steady state. Say that a k length run of zeroes ends at time m if

$$X_{m-k-1} \neq 0, \quad X_{m-k} = X_{m-k+1} = \ldots = X_{m-1} = 0, X_m \neq 0$$

Show that the probability of this event is $\pi_0(P_{0,0})^{k-1}(1 - P_{0,0})^2$, where π_0 is the limiting probability of state 0.

49. Consider a Markov chain with states 1, 2, 3 having transition probability matrix

$$\begin{pmatrix} .5 & .3 & .2 \\ 0 & .4 & .6 \\ .8 & 0 & .2 \end{pmatrix}$$

(a) If the chain is currently in state 1, find the probability that after two transitions it will be in state 2.

(b) Suppose you receive a reward $r(i) = i^2$ whenever the Markov chain is in state i, $i = 1, 2, 3$. Find your long run average reward per unit time.

Let N_i denote the number of transitions, starting in state i, until the Markov chain enters state 3.

(c) Find $E[N_1]$.

(d) Find $P(N_1 \leq 4)$.

(e) Find $P(N_1 = 4)$.

50. A Markov chain with states $1, \ldots, 6$ has transition probability matrix

$$\begin{pmatrix} .2 & .4 & 0 & .3 & 0 & .1 \\ .1 & .3 & 0 & .4 & 0 & .2 \\ 0 & 0 & .3 & .7 & 0 & 0 \\ 0 & 0 & .6 & .4 & 0 & 0 \\ 0 & 0 & 0 & 0 & .5 & .5 \\ 0 & 0 & 0 & 0 & .2 & .8 \end{pmatrix}$$

(a) Give the classes and tell which are recurrent and which are transient.

(b) Find $\lim_{n\to\infty} P^n_{1,2}$.

(c) Find $\lim_{n\to\infty} P^n_{5,6}$.

(d) Find $\lim_{n\to\infty} P^n_{1,3}$.

51. In Example 4.3, Gary is in a cheerful mood today. Find the expected number of days until he has been glum for three consecutive days.

52. A taxi driver provides service in two zones of a city. Fares picked up in zone A will have destinations in zone A with probability 0.6 or in zone B with probability 0.4. Fares picked up in zone B will have destinations in zone A with probability 0.3 or in zone B with probability 0.7. The driver's expected profit for a trip entirely in zone A is 6; for a trip entirely in zone B is 8; and for a trip that involves both zones is 12. Find the taxi driver's average profit per trip.

53. Find the average premium received per policyholder of the insurance company of Example 4.29 if $\lambda = 1/4$ for one-third of its clients, and $\lambda = 1/2$ for two-thirds of its clients.

54. Consider the Ehrenfest urn model in which M molecules are distributed between two urns, and at each time point one of the molecules is chosen at random and is then removed from its urn and placed in the other one. Let X_n denote the number of molecules in urn 1 after the nth switch and let $\mu_n = E[X_n]$. Show that
   (a) $\mu_{n+1} = 1 + (1 - 2/M)\mu_n$.
   (b) Use (a) to prove that

$$\mu_n = \frac{M}{2} + \left(\frac{M-2}{M}\right)^n \left(E[X_0] - \frac{M}{2}\right)$$

55. Consider a population of individuals each of whom possesses two genes that can be either type A or type a. Suppose that in outward appearance type A is dominant and type a is recessive. (That is, an individual will have only the outward characteristics of the recessive gene if its pair is aa.) Suppose that the population has stabilized, and the percentages of individuals having respective gene pairs AA, aa, and Aa are p, q, and r. Call an individual dominant or recessive depending on the outward characteristics it exhibits. Let S_{11} denote the probability that an offspring of two dominant parents will be recessive; and let S_{10} denote the probability that the offspring of one dominant and one recessive parent will be recessive. Compute S_{11} and S_{10} to show that $S_{11} = S_{10}^2$. (The quantities S_{10} and S_{11} are known in the genetics literature as *Snyder's ratios.*)

56. Suppose that on each play of the game a gambler either wins 1 with probability p or loses 1 with probability $1 - p$. The gambler continues betting until she or he is either up n or down m. What is the probability that the gambler quits a winner?

57. A particle moves among $n + 1$ vertices that are situated on a circle in the following manner. At each step it moves one step either in the clockwise direction with probability p or the counterclockwise direction with probability $q = 1 - p$. Starting at a specified state, call it state 0, let T be the time of the first return to state 0. Find the probability that all states have been visited by time T.

   Hint: Condition on the initial transition and then use results from the gambler's ruin problem.

58. In the gambler's ruin problem of Section 4.5.1, suppose the gambler's fortune is presently i, and suppose that we know that the gambler's fortune will eventually reach N (before it goes to 0). Given this information, show that the

probability he wins the next gamble is

$$\frac{p[1-(q/p)^{i+1}]}{1-(q/p)^i}, \qquad \text{if } p \neq \tfrac{1}{2}$$
$$\frac{i+1}{2i}, \qquad \text{if } p = \tfrac{1}{2}$$

Hint: The probability we want is

$$P\{X_{n+1}=i+1|X_n=i, \lim_{m\to\infty} X_m = N\}$$
$$= \frac{P\{X_{n+1}=i+1, \lim_m X_m = N|X_n=i\}}{P\{\lim_m X_m = N|X_n=i\}}$$

59. For the gambler's ruin model of Section 4.5.1, let M_i denote the mean number of games that must be played until the gambler either goes broke or reaches a fortune of N, given that he starts with i, $i=0,1,\ldots,N$. Show that M_i satisfies

$$M_0 = M_N = 0; \quad M_i = 1 + pM_{i+1} + qM_{i-1}, \quad i = 1,\ldots,N-1$$

Solve these equations to obtain

$$M_i = i(N-i), \qquad \text{if } p = \tfrac{1}{2}$$
$$= \frac{i}{q-p} - \frac{N}{q-p}\frac{1-(q/p)^i}{1-(q/p)^N}, \qquad \text{if } p \neq \tfrac{1}{2}$$

60. The following is the transition probability matrix of a Markov chain with states 1, 2, 3, 4

$$\mathbf{P} = \begin{pmatrix} .4 & .3 & .2 & .1 \\ .2 & .2 & .2 & .4 \\ .25 & .25 & .5 & 0 \\ .2 & .1 & .4 & .3 \end{pmatrix}$$

If $X_0 = 1$

(a) find the probability that state 3 is entered before state 4;

(b) find the mean number of transitions until either state 3 or state 4 is entered.

61. Suppose in the gambler's ruin problem that the probability of winning a bet depends on the gambler's present fortune. Specifically, suppose that α_i is the probability that the gambler wins a bet when his or her fortune is i. Given that the gambler's initial fortune is i, let $P(i)$ denote the probability that the gambler's fortune reaches N before 0.

(a) Derive a formula that relates $P(i)$ to $P(i-1)$ and $P(i+1)$.

(b) Using the same approach as in the gambler's ruin problem, solve the equation of part (a) for $P(i)$.

(c) Suppose that i balls are initially in urn 1 and $N-i$ are in urn 2, and suppose that at each stage one of the N balls is randomly chosen, taken from whichever urn it is in, and placed in the other urn. Find the probability that the first urn becomes empty before the second.

*62. Consider the particle from Exercise 57. What is the expected number of steps the particle takes to return to the starting position? What is the probability that all other positions are visited before the particle returns to its starting state?

63. For the Markov chain with states 1, 2, 3, 4 whose transition probability matrix $\mathbf{P}$ is as specified below find f_{i3} and s_{i3} for $i = 1, 2, 3$.

$$\mathbf{P} = \begin{bmatrix} 0.4 & 0.2 & 0.1 & 0.3 \\ 0.1 & 0.5 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.2 & 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

64. Consider a branching process having $\mu < 1$. Show that if $X_0 = 1$, then the expected number of individuals that ever exist in this population is given by $1/(1-\mu)$. What if $X_0 = n$?

65. In a branching process having $X_0 = 1$ and $\mu > 1$, prove that π_0 is the *smallest* positive number satisfying Eq. (4.20).

Hint: Let π be any solution of $\pi = \sum_{j=0}^{\infty} \pi^j P_j$. Show by mathematical induction that $\pi \geqslant P\{X_n = 0\}$ for all n, and let $n \to \infty$. In using the induction argue that

$$P\{X_n = 0\} = \sum_{j=0}^{\infty} (P\{X_{n-1} = 0\})^j P_j$$

66. For a branching process, calculate π_0 when
   (a) $P_0 = \frac{1}{4}, P_2 = \frac{3}{4}$.
   (b) $P_0 = \frac{1}{4}, P_1 = \frac{1}{2}, P_2 = \frac{1}{4}$.
   (c) $P_0 = \frac{1}{6}, P_1 = \frac{1}{2}, P_3 = \frac{1}{3}$.

67. At all times, an urn contains N balls—some white balls and some black balls. At each stage, a coin having probability $p, 0 < p < 1$, of landing heads is flipped. If heads appears, then a ball is chosen at random from the urn and is replaced by a white ball; if tails appears, then a ball is chosen from the urn and is replaced by a black ball. Let X_n denote the number of white balls in the urn after the nth stage.
   (a) Is $\{X_n, n \geqslant 0\}$ a Markov chain? If so, explain why.
   (b) What are its classes? What are their periods? Are they transient or recurrent?
   (c) Compute the transition probabilities P_{ij}.
   (d) Let $N = 2$. Find the proportion of time in each state.
   (e) Based on your answer in part (d) and your intuition, guess the answer for the limiting probability in the general case.

(f) Prove your guess in part (e) either by showing that Theorem (4.1) is satisfied or by using the results of Example 4.37.

(g) If $p = 1$, what is the expected time until there are only white balls in the urn if initially there are i white and $N - i$ black?

***68.** (a) Show that the limiting probabilities of the reversed Markov chain are the same as for the forward chain by showing that they satisfy the equations

$$\pi_j = \sum_i \pi_i Q_{ij}$$

(b) Give an intuitive explanation for the result of part (a).

69. M balls are initially distributed among m urns. At each stage one of the balls is selected at random, taken from whichever urn it is in, and then placed, at random, in one of the other $m - 1$ urns. Consider the Markov chain whose state at any time is the vector $(n_1, \ldots, n_m)$ where n_i denotes the number of balls in urn i. Guess at the limiting probabilities for this Markov chain and then verify your guess and show at the same time that the Markov chain is time reversible.

70. A total of m white and m black balls are distributed among two urns, with each urn containing m balls. At each stage, a ball is randomly selected from each urn and the two selected balls are interchanged. Let X_n denote the number of black balls in urn 1 after the nth interchange.

(a) Give the transition probabilities of the Markov chain $X_n, n \geqslant 0$.

(b) Without any computations, what do you think are the limiting probabilities of this chain?

(c) Find the limiting probabilities and show that the stationary chain is time reversible.

71. It follows from Theorem 4.2 that for a time reversible Markov chain

$$P_{ij}P_{jk}P_{ki} = P_{ik}P_{kj}P_{ji}, \quad \text{for all } i, j, k$$

It turns out that if the state space is finite and $P_{ij} > 0$ for all i, j, then the preceding is also a sufficient condition for time reversibility. (That is, in this case, we need only check Eq. (4.26) for paths from i to i that have only two intermediate states.) Prove this.

Hint: Fix i and show that the equations

$$\pi_j P_{jk} = \pi_k P_{kj}$$

are satisfied by $\pi_j = cP_{ij}/P_{ji}$, where c is chosen so that $\sum_j \pi_j = 1$.

72. For a time reversible Markov chain, argue that the rate at which transitions from i to j to k occur must equal the rate at which transitions from k to j to i occur.

73. There are k players, with player i having value $v_i > 0$, $i = 1, \ldots, k$. In every period, two of the players play a game. Whoever wins then plays the next game against a randomly chosen one of the other $k - 1$ players (including the one who has just lost). Suppose that whenever i and j play, i wins with probability $\frac{v_i}{v_i+v_j}$. Let X_n denote the winner of game n.

(a) Give the transition probabilities of the Markov chain $\{X_n, n \geq 1\}$.
(b) Give the stationarity equations that are uniquely satisfied by the π_j.
(c) Give the time reversibility equations.
(d) Find the proportion of all games that are won by j, $j = 1, \ldots, k$.
(e) Find the proportion of games that involve player j as one of the contestants.

74. A group of n processors is arranged in an ordered list. When a job arrives, the first processor in line attempts it; if it is unsuccessful, then the next in line tries it; if it too is unsuccessful, then the next in line tries it, and so on. When the job is successfully processed or after all processors have been unsuccessful, the job leaves the system. At this point we are allowed to reorder the processors, and a new job appears. Suppose that we use the one-closer reordering rule, which moves the processor that was successful one closer to the front of the line by interchanging its position with the one in front of it. If all processors were unsuccessful (or if the processor in the first position was successful), then the ordering remains the same. Suppose that each time processor i attempts a job then, independently of anything else, it is successful with probability p_i.
(a) Define an appropriate Markov chain to analyze this model.
(b) Show that this Markov chain is time reversible.
(c) Find the long-run probabilities.

75. A Markov chain is said to be a tree process if
(i) $P_{ij} > 0$ whenever $P_{ji} > 0$,
(ii) for every pair of states i and j, $i \neq j$, there is a unique sequence of distinct states $i = i_0, i_1, \ldots, i_{n-1}, i_n = j$ such that

$$P_{i_k, i_{k+1}} > 0, \quad k = 0, 1, \ldots, n-1$$

In other words, a Markov chain is a tree process if for every pair of distinct states i and j there is a unique way for the process to go from i to j without reentering a state (and this path is the reverse of the unique path from j to i). Argue that an ergodic tree process is time reversible.

76. On a chessboard compute the expected number of plays it takes a knight, starting in one of the four corners of the chessboard, to return to its initial position if we assume that at each play it is equally likely to choose any of its legal moves. (No other pieces are on the board.)

Hint: Make use of Example 4.38.

77. In a Markov decision problem, another criterion often used, different than the expected average return per unit time, is that of the expected discounted return. In this criterion we choose a number α, $0 < \alpha < 1$, and try to choose a policy so as to maximize $E[\sum_{i=0}^{\infty} \alpha^i R(X_i, a_i)]$ (that is, rewards at time n are discounted at rate α^n). Suppose that the initial state is chosen according to the probabilities b_i. That is,

$$P\{X_0 = i\} = b_i, \quad i = 1, \ldots, n$$

For a given policy $\boldsymbol{\beta}$ let y_{ja} denote the expected discounted time that the process is in state j and action a is chosen. That is,

$$y_{ja} = E_{\boldsymbol{\beta}}\left[\sum_{n=0}^{\infty} \alpha^n I_{\{X_n=j, a_n=a\}}\right]$$

where for any event A the indicator variable I_A is defined by

$$I_A = \begin{cases} 1, & \text{if } A \text{ occurs} \\ 0, & \text{otherwise} \end{cases}$$

(a) Show that

$$\sum_a y_{ja} = E\left[\sum_{n=0}^{\infty} \alpha^n I_{\{X_n=j\}}\right]$$

or, in other words, $\sum_a y_{ja}$ is the expected discounted time in state j under $\boldsymbol{\beta}$.

(b) Show that

$$\sum_j \sum_a y_{ja} = \frac{1}{1-\alpha},$$

$$\sum_a y_{ja} = b_j + \alpha \sum_i \sum_a y_{ia} P_{ij}(a)$$

Hint: For the second equation, use the identity

$$I_{\{X_{n+1}=j\}} = \sum_i \sum_a I_{\{X_n=i, a_n=a\}} I_{\{X_{n+1}=j\}}$$

Take expectations of the preceding to obtain

$$E\left[I_{X_{n+1}=j\}}\right] = \sum_i \sum_a E\left[I_{\{X_n=i, a_n=a\}}\right] P_{ij}(a)$$

(c) Let $\{y_{ja}\}$ be a set of numbers satisfying

$$\sum_j \sum_a y_{ja} = \frac{1}{1-\alpha}, \qquad (4.38)$$

$$\sum_a y_{ja} = b_j + \alpha \sum_i \sum_a y_{ia} P_{ij}(a)$$

Argue that y_{ja} can be interpreted as the expected discounted time that the process is in state j and action a is chosen when the initial state is chosen according to the probabilities b_j and the policy $\boldsymbol{\beta}$, given by

$$\beta_i(a) = \frac{y_{ia}}{\sum_a y_{ia}}$$

is employed.

Hint: Derive a set of equations for the expected discounted times when policy $\boldsymbol{\beta}$ is used and show that they are equivalent to Eq. (4.38).

(d) Argue that an optimal policy with respect to the expected discounted return criterion can be obtained by first solving the linear program

$$\begin{aligned} \text{maximize} \quad & \sum_j \sum_a y_{ja} R(j,a), \\ \text{such that} \quad & \sum_j \sum_a y_{ja} = \frac{1}{1-\alpha}, \\ & \sum_a y_{ja} = b_j + \alpha \sum_i \sum_a y_{ia} P_{ij}(a), \\ & y_{ja} \geqslant 0, \quad \text{all } j, a; \end{aligned}$$

and then defining the policy β^* by

$$\beta_i^*(a) = \frac{y_{ia}^*}{\sum_a y_{ia}^*}$$

where the y_{ja}^* are the solutions of the linear program.

78. For the Markov chain of Exercise 5, suppose that $p(s|j)$ is the probability that signal s is emitted when the underlying Markov chain state is j, $j = 0, 1, 2$.

(a) What proportion of emissions are signal s?

(b) What proportion of those times in which signal s is emitted is 0 the underlying state?

79. In Example 4.45, what is the probability that the first 4 items produced are all acceptable?

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