Now if we consider an arbitrary arrival, then we have the following identity:

$$\text{work in system when customer arrives} = k \times \text{time customer spends in queue} + R \tag{8.62}$$

where R is the sum of the remaining service times of all other customers in service at the moment when our arrival enters service.

The foregoing follows because while the arrival is waiting in queue, work is being processed at a rate k per unit time (since all servers are busy). Thus, an amount of work $k \times$ time in queue is processed while he waits in queue. Now, all of this work was present when he arrived and in addition the remaining work on those still being served when he enters service was also present when he arrived—so we obtain Eq. (8.62). For an illustration, suppose that there are three servers all of whom are busy when the customer arrives. Suppose, in addition, that there are no other customers in the system and also that the remaining service times of the three people in service are 3, 6, and 7. Hence, the work seen by the arrival is $3+6+7=16$. Now the arrival will spend 3 time units in queue, and at the moment he enters service, the remaining times of the other two customers are $6-3=3$ and $7-3=4$. Hence, $R=3+4=7$ and as a check of Eq. (8.62) we see that $16=3\times 3+7$.

Taking expectations of Eq. (8.62) and using the fact that Poisson arrivals see time averages, we obtain

$$V = kW_Q + E[R]$$

which, along with Eq. (8.61), would enable us to solve for W_Q if we could compute $E[R]$. However there is no known method for computing $E[R]$ and in fact, there is no known exact formula for W_Q. The following approximation for W_Q was obtained in Reference 6 by using the foregoing approach and then approximating $E[R]$:

$$W_Q \approx \frac{\lambda^k E[S^2](E[S])^{k-1}}{2(k-1)!(k-\lambda E[S])^2\left[\sum_{n=0}^{k-1}\frac{(\lambda E[S])^n}{n!} + \frac{(\lambda E[S])^k}{(k-1)!(k-\lambda E[S])}\right]} \tag{8.63}$$

The preceding approximation has been shown to be quite close to W_Q when the service distribution is gamma. It is also exact when G is exponential.

Exercises

1. For the $M/M/1$ queue, compute
   (a) the expected number of arrivals during a service period and
   (b) the probability that no customers arrive during a service period.

   Hint: "Condition."

*2. Machines in a factory break down at an exponential rate of six per hour. There is a single repairman who fixes machines at an exponential rate of eight per hour. The cost incurred in lost production when machines are out of service is $10 per hour per machine. What is the average cost rate incurred due to failed machines?

3. The manager of a market can hire either Mary or Alice. Mary, who gives service at an exponential rate of 20 customers per hour, can be hired at a rate of $3 per hour. Alice, who gives service at an exponential rate of 30 customers per hour, can be hired at a rate of $C per hour. The manager estimates that, on the average, each customer's time is worth $1 per hour and should be accounted for in the model. Assume customers arrive at a Poisson rate of 10 per hour

   (a) What is the average cost per hour if Mary is hired? If Alice is hired?

   (b) Find C if the average cost per hour is the same for Mary and Alice.

4. In the $M/M/1$ system, derive P_0 by equating the rate at which customers arrive with the rate at which they depart.

5. Suppose customers arrive to a two server system according to a Poisson process with rate λ, and suppose that each arrival is, independently, sent either to server 1 with probability α or to server 2 with probability $1-\alpha$. Suppose the service time at server i is exponential with rate μ_i, $i = 1, 2$.

   (a) Find $W(\alpha)$, the average amount of time a customer spends in the system.

   (b) If $\lambda = 1$ and $\mu_i = i$, $i = 1, 2$, find the value of α that minimizes $W(\alpha)$.

6. Suppose that a customer of the $M/M/1$ system spends the amount of time $x > 0$ waiting in queue before entering service.

   (a) Show that, conditional on the preceding, the number of other customers that were in the system when the customer arrived is distributed as $1 + P$, where P is a Poisson random variable with mean λ.

   (b) Let W_Q^* denote the amount of time that an $M/M/1$ customer spends in queue. As a by-product of your analysis in part (a), show that

$$P\{W_Q^* \le x\} = \begin{cases} 1 - \frac{\lambda}{\mu} & \text{if } x = 0 \\ 1 - \frac{\lambda}{\mu} + \frac{\lambda}{\mu}(1 - e^{-(\mu-\lambda)x}) & \text{if } x > 0 \end{cases}$$

7. It follows from Exercise 6 that if, in the $M/M/1$ model, W_Q^* is the amount of time that a customer spends waiting in queue, then

$$W_Q^* = \begin{cases} 0, & \text{with probability } 1 - \lambda/\mu \\ \text{Exp}(\mu - \lambda), & \text{with probability } \lambda/\mu \end{cases}$$

   where $\text{Exp}(\mu - \lambda)$ is an exponential random variable with rate $\mu - \lambda$. Using this, find $\text{Var}(W_Q^*)$.

*8. Show that W is smaller in an $M/M/1$ model having arrivals at rate λ and service at rate 2μ than it is in a two-server $M/M/2$ model with arrivals at rate λ and with each server at rate μ. Can you give an intuitive explanation for this result? Would it also be true for W_Q?

9. Consider the $M/M/1$ queue with impatient customers model as presented in Example 8.9. Give your answers in terms of the limiting probabilities $P_n, n \geq 0$.

(a) What is the average amount of time that a customer spends in queue.

(b) If e_n denotes the probability that a customer who finds n others in the system upon arrival will be served, find $e_n, n \geq 0$.

(c) Find the conditional probability that a served customer found n in the system upon arrival. That is, find $P(\text{arrival finds } n | \text{ arrival is served})$.

(d) Find the average amount of time spent in queue by a customer that is served.

(e) Find the average amount of time spent in queue by a customer that departs before entering service.

10. A facility produces items according to a Poisson process with rate λ. However, it has shelf space for only k items and so it shuts down production whenever k items are present. Customers arrive at the facility according to a Poisson process with rate μ. Each customer wants one item and will immediately depart either with the item or empty handed if there is no item available.

(a) Find the proportion of customers that go away empty handed.

(b) Find the average time that an item is on the shelf.

(c) Find the average number of items on the shelf.

11. A group of n customers moves around among two servers. Upon completion of service, the served customer then joins the queue (or enters service if the server is free) at the other server. All service times are exponential with rate μ. Find the proportion of time that there are j customers at server 1, $j = 0, \ldots, n$.

12. A group of m customers frequents a single-server station in the following manner. When a customer arrives, he or she either enters service if the server is free or joins the queue otherwise. Upon completing service the customer departs the system, but then returns after an exponential time with rate θ. All service times are exponentially distributed with rate μ.

(a) Find the average rate at which customers enter the station.

(b) Find the average time that a customer spends in the station per visit.

***13.** Families arrive at a taxi stand according to a Poisson process with rate λ. An arriving family finding N other families waiting for a taxi does not wait. Taxis arrive at the taxi stand according to a Poisson process with rate μ. A taxi finding M other taxis waiting does not wait. Derive expressions for the following quantities.

(a) The proportion of time there are no families waiting.

(b) The proportion of time there are no taxis waiting.

(c) The average amount of time that a family waits.

(d) The average amount of time that a taxi waits.

(e) The fraction of families that take taxis.

Now redo the problem if we assume that $N = M = \infty$ and that each family will only wait for an exponential time with rate α before seeking other transportation, and each taxi will only wait for an exponential time with rate β before departing without a fare.

14. Customers arrive to a single server system in accordance with a Poisson process with rate λ. Arrivals only enter if the server is free. Each customer is either a type 1 customer with probability p or a type 2 customer with probability $1-p$. The time it takes to serve a type i customer is exponential with rate μ_i, $i=1,2$. Find the average amount of time an entering customer spends in the system.

15. Customers arrive to a two server system in accordance with a Poisson process with rate λ. Server 1 is the preferred server, and an arrival finding server 1 free enters service with 1; an arrival finding 1 busy but 2 free, enters service with 2. Arrivals finding both servers busy do not enter. A customer who is with server 2 at a moment when server 1 becomes free, immediately leaves server 2 and moves over to server 1. After completing a service (with either server) the customer departs. The service times at server i are exponential with rate μ_i, $i=1,2$.
    - **(a)** Define states and give the transition diagram.
    - **(b)** Find the long run proportion of time the system is in each state.
    - **(c)** Find the proportion of all arrivals that enter the system.
    - **(d)** Find the average time that an entering customer spends in the system.
    - **(e)** Find the proportion of entering customers that complete service with server 2.

16. Consider a 2 server system where customers arrive according to a Poisson process with rate λ, and where each arrival is sent to the server currently having the shortest queue. (If they have the same length queue then the choice is made at random.) The service time at either server is exponential with rate μ, where $\lambda < 2\mu$. For $n \geq 0$, say that the state is (n, n) if both servers currently have n customers, and say that the state is (n, m), $n < m$, if one of the servers has n customers and the other has m.
    - **(a)** Write down the balance equation equating the rate at which the process enters and leaves a state for state $(0, 0)$.
    - **(b)** Write down the balance equations equating the rate at which the process enters and leaves states of the form $(0, m)$, $m > 0$.
    - **(c)** Write down the balance equations for the states (n, n), $n > 0$.
    - **(d)** Write down the balance equations for the states (n, m), $0 < n < m$.
    - **(e)** In terms of the solution of the balance equations, find the average time a customer spends in the system.

17. Two customers move about among three servers. Upon completion of service at server i, the customer leaves that server and enters service at whichever of the other two servers is free. (Therefore, there are always two busy servers.) If the service times at server i are exponential with rate μ_i, $i = 1, 2, 3$, what proportion of time is server i idle?

18. Consider a queueing system having two servers and no queue. There are two types of customers. Type 1 customers arrive according to a Poisson process having rate λ_1, and will enter the system if either server is free. The service time of a type 1 customer is exponential with rate μ_1. Type 2 customers arrive according to a Poisson process having rate λ_2. A type 2 customer requires the

simultaneous use of both servers; hence, a type 2 arrival will only enter the system if both servers are free. The time that it takes (the two servers) to serve a type 2 customer is exponential with rate μ_2. Once a service is completed on a customer, that customer departs the system.

(a) Define states to analyze the preceding model.
(b) Give the balance equations.
In terms of the solution of the balance equations, find
(c) the average amount of time an entering customer spends in the system;
(d) the fraction of served customers that are type 1.

19. Consider a sequential-service system consisting of two servers, A and B. Arriving customers will enter this system only if server A is free. If a customer does enter, then he is immediately served by server A. When his service by A is completed, he then goes to B if B is free, or if B is busy, he leaves the system. Upon completion of service at server B, the customer departs. Assume that the (Poisson) arrival rate is two customers an hour, and that A and B serve at respective (exponential) rates of four and two customers an hour.
(a) What proportion of customers enter the system?
(b) What proportion of entering customers receive service from B?
(c) What is the average number of customers in the system?
(d) What is the average amount of time that an entering customer spends in the system?

20. Customers arrive at a two-server system according to a Poisson process having rate $\lambda = 5$. An arrival finding server 1 free will begin service with that server. An arrival finding server 1 busy and server 2 free will enter service with server 2. An arrival finding both servers busy goes away. Once a customer is served by either server, he departs the system. The service times at server i are exponential with rates μ_i, where $\mu_1 = 4$, $\mu_2 = 2$.
(a) What is the average time an entering customer spends in the system?
(b) What proportion of time is server 2 busy?

21. Customers arrive at a two-server station in accordance with a Poisson process with a rate of two per hour. Arrivals finding server 1 free begin service with that server. Arrivals finding server 1 busy and server 2 free begin service with server 2. Arrivals finding both servers busy are lost. When a customer is served by server 1, she then either enters service with server 2 if 2 is free or departs the system if 2 is busy. A customer completing service at server 2 departs the system. The service times at server 1 and server 2 are exponential random variables with respective rates of four and six per hour.
(a) What fraction of customers do not enter the system?
(b) What is the average amount of time that an entering customer spends in the system?
(c) What fraction of entering customers receives service from server 1?

22. Arrivals to a three-server system are according to a Poisson process with rate λ. Arrivals finding server 1 free enter service with 1. Arrivals finding 1 busy but 2 free enter service with 2. Arrivals finding both 1 and 2 busy do not join the system. After completion of service at either 1 or 2 the customer will then either go to server 3 if 3 is free or depart the system if 3 is busy. After service at 3 customers depart the system. The service times at i are exponential with rate μ_i, $i = 1, 2, 3$.

(a) Define states to analyze the above system.

(b) Give the balance equations.

(c) In terms of the solution of the balance equations, what is the average time that an entering customer spends in the system?

(d) Find the probability that a customer who arrives when the system is empty is served by server 3.

23. The economy alternates between good and bad periods. During good times customers arrive at a certain single-server queueing system in accordance with a Poisson process with rate λ_1, and during bad times they arrive in accordance with a Poisson process with rate λ_2. A good time period lasts for an exponentially distributed time with rate α_1, and a bad time period lasts for an exponential time with rate α_2. An arriving customer will only enter the queueing system if the server is free; an arrival finding the server busy goes away. All service times are exponential with rate μ.

(a) Define states so as to be able to analyze this system.

(b) Give a set of linear equations whose solution will yield the long-run proportion of time the system is in each state.

In terms of the solutions of the equations in part (b),

(c) what proportion of time is the system empty?

(d) what is the average rate at which customers enter the system?

24. There are two types of customers. Type 1 and 2 customers arrive in accordance with independent Poisson processes with respective rate λ_1 and λ_2. There are two servers. A type 1 arrival will enter service with server 1 if that server is free; if server 1 is busy and server 2 is free, then the type 1 arrival will enter service with server 2. If both servers are busy, then the type 1 arrival will go away. A type 2 customer can only be served by server 2; if server 2 is free when a type 2 customer arrives, then the customer enters service with that server. If server 2 is busy when a type 2 arrives, then that customer goes away. Once a customer is served by either server, he departs the system. Service times at server i are exponential with rate μ_i, $i = 1, 2$.

Suppose we want to find the average number of customers in the system.

(a) Define states.

(b) Give the balance equations. Do not attempt to solve them.

In terms of the long-run probabilities, what is

(c) the average number of customers in the system?

(d) the average time a customer spends in the system?

*25. Suppose in Exercise 24 we want to find out the proportion of time there is a type 1 customer with server 2. In terms of the long-run probabilities given in Exercise 24, what is

(a) the rate at which a type 1 customer enters service with server 2?

(b) the rate at which a type 2 customer enters service with server 2?

(c) the fraction of server 2's customers that are type 1?

(d) the proportion of time that a type 1 customer is with server 2?

26. Customers arrive at a single-server station in accordance with a Poisson process with rate λ. All arrivals that find the server free immediately enter service. All service times are exponentially distributed with rate μ. An arrival that finds the server busy will leave the system and roam around "in orbit" for an exponential time with rate θ at which time it will then return. If the server is busy when an orbiting customer returns, then that customer returns to orbit for another exponential time with rate θ before returning again. An arrival that finds the server busy and N other customers in orbit will depart and not return. That is, N is the maximum number of customers in orbit.

(a) Define states.
(b) Give the balance equations.
In terms of the solution of the balance equations, find
(c) the proportion of all customers that are eventually served;
(d) the average time that a served customer spends waiting in orbit.

27. Consider the $M/M/1$ system in which customers arrive at rate λ and the server serves at rate μ. However, suppose that in any interval of length h in which the server is busy there is a probability $\alpha h + o(h)$ that the server will experience a breakdown, which causes the system to shut down. All customers that are in the system depart, and no additional arrivals are allowed to enter until the breakdown is fixed. The time to fix a breakdown is exponentially distributed with rate β.

(a) Define appropriate states.
(b) Give the balance equations.
In terms of the long-run probabilities,
(c) what is the average amount of time that an entering customer spends in the system?
(d) what proportion of entering customers complete their service?
(e) what proportion of customers arrive during a breakdown?

***28.** Reconsider Exercise 27, but this time suppose that a customer that is in the system when a breakdown occurs remains there while the server is being fixed. In addition, suppose that new arrivals during a breakdown period are allowed to enter the system. What is the average time a customer spends in the system?

29. Poisson (λ) arrivals join a queue in front of two parallel servers A and B, having exponential service rates μ_A and μ_B (see Fig. 8.4). When the system is empty, arrivals go into server A with probability α and into B with probability $1 - \alpha$. Otherwise, the head of the queue takes the first free server.

(a) Define states and set up the balance equations. Do not solve.
(b) In terms of the probabilities in part (a), what is the average number in the system? Average number of servers idle?

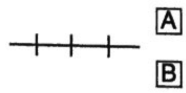

Figure 8.4

(c) In terms of the probabilities in part (a), what is the probability that an arbitrary arrival will get serviced in A?

30. In a queue with unlimited waiting space, arrivals are Poisson (parameter λ) and service times are exponentially distributed (parameter μ). However, the server waits until K people are present before beginning service on the first customer; thereafter, he services one at a time until all K units, and all subsequent arrivals, are serviced. The server is then "idle" until K new arrivals have occurred.
    (a) Define an appropriate state space, draw the transition diagram, and set up the balance equations.
    (b) In terms of the limiting probabilities, what is the average time a customer spends in queue?
    (c) What conditions on λ and μ are necessary?

31. Consider a single-server exponential system in which ordinary customers arrive at a rate λ and have service rate μ. In addition, there is a special customer who has a service rate μ_1. Whenever this special customer arrives, she goes directly into service (if anyone else is in service, then this person is bumped back into queue). When the special customer is not being serviced, she spends an exponential amount of time (with mean $1/\theta$) out of the system.
    (a) What is the average arrival rate of the special customer?
    (b) Define an appropriate state space and set up balance equations.
    (c) Find the probability that an ordinary customer is bumped n times.

*32. Let D denote the time between successive departures in a stationary $M/M/1$ queue with $\lambda < \mu$. Show, by conditioning on whether or not a departure has left the system empty, that D is exponential with rate λ.

Hint: By conditioning on whether or not the departure has left the system empty we see that

$$D = \begin{cases} \text{Exponential}(\mu), & \text{with probability } \lambda/\mu \\ \text{Exponential}(\lambda) * \text{Exponential}(\mu), & \text{with probability } 1 - \lambda/\mu \end{cases}$$

where Exponential(λ) $*$ Exponential(μ) represents the sum of two independent exponential random variables having rates μ and λ. Now use moment-generating functions to show that D has the required distribution.

Note that the preceding does not prove that the departure process is Poisson. To prove this we need show not only that the interdeparture times are all exponential with rate λ, but also that they are independent.

33. Potential customers arrive to a single-server hair salon according to a Poisson process with rate λ. A potential customer who finds the server free enters the system; a potential customer who finds the server busy goes away. Each potential customer is type i with probability p_i, where $p_1 + p_2 + p_3 = 1$. Type 1 customers have their hair washed by the server; type 2 customers have their hair cut by the server; and type 3 customers have their hair first washed and then cut by the server. The time that it takes the server to wash hair is exponentially distributed with rate μ_1, and the time that it takes the server to cut hair is exponentially distributed with rate μ_2.

(a) Explain how this system can be analyzed with four states.
(b) Give the equations whose solution yields the proportion of time the system is in each state.
In terms of the solution of the equations of (b), find
(c) the proportion of time the server is cutting hair;
(d) the average arrival rate of entering customers.

34. For the tandem queue model verify that

$$P_{n,m} = (\lambda/\mu_1)^n(1-\lambda/\mu_1)(\lambda/\mu_2)^m(1-\lambda/\mu_2)$$

satisfies the balance Eqs. (8.15).

35. Consider a network of three stations with a single server at each station. Customers arrive at stations 1, 2, 3 in accordance with Poisson processes having respective rates 5, 10, and 15. The service times at the three stations are exponential with respective rates 10, 50, and 100. A customer completing service at station 1 is equally likely to (i) go to station 2, (ii) go to station 3, or (iii) leave the system. A customer departing service at station 2 always goes to station 3. A departure from service at station 3 is equally likely to either go to station 2 or leave the system.
(a) What is the average number of customers in the system (consisting of all three stations)?
(b) What is the average time a customer spends in the system?

36. Consider a closed queueing network consisting of two customers moving among two servers, and suppose that after each service completion the customer is equally likely to go to either server—that is, $P_{1,2} = P_{2,1} = \frac{1}{2}$. Let μ_i denote the exponential service rate at server i, $i = 1, 2$.
(a) Determine the average number of customers at each server.
(b) Determine the service completion rate for each server.

37. Explain how a Markov chain Monte Carlo simulation using the Gibbs sampler can be utilized to estimate
(a) the distribution of the amount of time spent at server j on a visit.

Hint: Use the arrival theorem.

(b) the proportion of time a customer is with server j (i.e., either in server j's queue or in service with j).

38. For open queueing networks
(a) state and prove the equivalent of the arrival theorem;
(b) derive an expression for the average amount of time a customer spends waiting in queues.

39. Customers arrive at a single-server station in accordance with a Poisson process having rate λ. Each customer has a value. The successive values of customers are independent and come from a uniform distribution on (0, 1). The service time of a customer having value x is a random variable with mean $3 + 4x$ and variance 5.
(a) What is the average time a customer spends in the system?
(b) What is the average time a customer having value x spends in the system?

*40. Compare the $M/G/1$ system for first-come, first-served queue discipline with one of last-come, first-served (for instance, in which units for service are taken from the top of a stack). Would you think that the queue size, waiting time, and busy-period distribution differ? What about their means? What if the queue discipline was always to choose at random among those waiting? Intuitively, which discipline would result in the smallest variance in the waiting time distribution?

41. In an $M/G/1$ queue,
   (a) what proportion of departures leave behind 0 work?
   (b) what is the average work in the system as seen by a departure?

42. For the $M/G/1$ queue, let X_n denote the number in the system left behind by the nth departure.
   (a) If

$$X_{n+1} = \begin{cases} X_n - 1 + Y_n, & \text{if } X_n \geq 1 \\ Y_n, & \text{if } X_n = 0 \end{cases}$$

   what does Y_n represent?
   (b) Rewrite the preceding as

$$X_{n+1} = X_n - 1 + Y_n + \delta_n \tag{8.64}$$

   where

$$\delta_n = \begin{cases} 1, & \text{if } X_n = 0 \\ 0, & \text{if } X_n \geq 1 \end{cases}$$

   Take expectations and let $n \to \infty$ in Eq. (8.64) to obtain

$$E[\delta_\infty] = 1 - \lambda E[S]$$

   (c) Square both sides of Eq. (8.64), take expectations, and then let $n \to \infty$ to obtain

$$E[X_\infty] = \frac{\lambda^2 E[S^2]}{2(1 - \lambda E[S])} + \lambda E[S]$$

   (d) Argue that $E[X_\infty]$, the average number as seen by a departure, is equal to L.

*43. Consider an $M/G/1$ system in which the first customer in a busy period has the service distribution G_1 and all others have distribution G_2. Let C denote the number of customers in a busy period, and let S denote the service time of a customer chosen at random.

Argue that
   (a) $a_0 = P_0 = 1 - \lambda E[S]$.
   (b) $E[S] = a_0 E[S_1] + (1 - a_0) E[S_2]$ where S_i has distribution G_i.

(c) Use (a) and (b) to show that $E[B]$, the expected length of a busy period, is given by

$$E[B] = \frac{E[S_1]}{1 - \lambda E[S_2]}$$

(d) Find $E[C]$.

44. Consider a $M/G/1$ system with $\lambda E[S] < 1$.

(a) Suppose that service is about to begin at a moment when there are n customers in the system.

(i) Argue that the additional time until there are only $n-1$ customers in the system has the same distribution as a busy period.

(ii) What is the expected additional time until the system is empty?

(b) Suppose that the work in the system at some moment is A. We are interested in the expected additional time until the system is empty—call it $E[T]$. Let N denote the number of arrivals during the first A units of time.

(i) Compute $E[T|N]$.

(ii) Compute $E[T]$.

45. Carloads of customers arrive at a single-server station in accordance with a Poisson process with rate 4 per hour. The service times are exponentially distributed with rate 20 per hour. If each carload contains either 1, 2, or 3 customers with respective probabilities $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{1}{4}$, compute the average customer delay in queue.

46. In the two-class priority queueing model of Section 8.6.2, what is W_Q? Show that W_Q is less than it would be under FIFO if $E[S_1] < E[S_2]$ and greater than under FIFO if $E[S_1] > E[S_2]$.

47. In a two-class priority queueing model suppose that a cost of C_i per unit time is incurred for each type i customer that waits in queue, $i = 1, 2$. Show that type 1 customers should be given priority over type 2 (as opposed to the reverse) if

$$\frac{E[S_1]}{C_1} < \frac{E[S_2]}{C_2}$$

48. Consider the priority queueing model of Section 8.6.2 but now suppose that if a type 2 customer is being served when a type 1 arrives then the type 2 customer is bumped out of service. This is called the preemptive case. Suppose that when a bumped type 2 customer goes back in service his service begins at the point where it left off when he was bumped.

(a) Argue that the work in the system at any time is the same as in the nonpreemptive case.

(b) Derive W_Q^1.

Hint: How do type 2 customers affect type 1s?

(c) Why is it not true that

$$V_Q^2 = \lambda_2 E[S_2] W_Q^2$$

(d) Argue that the work seen by a type 2 arrival is the same as in the nonpreemptive case, and so

$$W_Q^2 = W_Q^2(\text{nonpreemptive}) + E[\text{extra time}]$$

where the extra time is due to the fact that he may be bumped.

(e) Let N denote the number of times a type 2 customer is bumped. Why is

$$E[\text{extra time}|N] = \frac{NE[S_1]}{1 - \lambda_1 E[S_1]}$$

Hint: When a type 2 is bumped, relate the time until he gets back in service to a "busy period."

(f) Let S_2 denote the service time of a type 2. What is $E[N|S_2]$?

(g) Combine the preceding to obtain

$$W_Q^2 = W_Q^2(\text{nonpreemptive}) + \frac{\lambda_1 E[S_1]E[S_2]}{1 - \lambda_1 E[S_1]}$$

*49. Calculate explicitly (not in terms of limiting probabilities) the average time a customer spends in the system in Exercise 28.

50. In the $G/M/1$ model if G is exponential with rate λ show that $\beta = \lambda/\mu$.

51. In the k server Erlang loss model, suppose that $\lambda = 1$ and $E[S] = 4$. Find L if $P_k = .2$.

52. Verify the formula given for the P_i of the $M/M/k$.

53. In the Erlang loss system suppose the Poisson arrival rate is $\lambda = 2$, and suppose there are three servers, each of whom has a service distribution that is uniformly distributed over $(0, 2)$. What proportion of potential customers is lost?

54. In the $M/M/k$ system,
(a) what is the probability that a customer will have to wait in queue?
(b) determine L and W.

55. Verify the formula for the distribution of W_Q^* given for the $G/M/k$ model.

*56. Consider a system where the interarrival times have an arbitrary distribution F, and there is a single server whose service distribution is G. Let D_n denote the amount of time the nth customer spends waiting in queue. Interpret S_n, T_n so that

$$D_{n+1} = \begin{cases} D_n + S_n - T_n, & \text{if } D_n + S_n - T_n \geq 0 \\ 0, & \text{if } D_n + S_n - T_n < 0 \end{cases}$$

57. Consider a model in which the interarrival times have an arbitrary distribution F, and there are k servers each having service distribution G. What condition on F and G do you think would be necessary for there to exist limiting probabilities?