Ross 12, Q59 ①

$M_0 = M_N = 0$. For $i = 1, \dots, N-1$

$M_i = E(T \mid X_0 = i)$, $T = \inf(n \geq 0 : X_n = 0 \text{ or } N)$

$= E\left(\sum_{j=0}^{N} E(T \mid X_0 = i, X_1 = j) \mid X_0 = i\right)$

$= E\left(\sum_{j=0}^{N} E(T \mid X_1 = j)\, P_{ij} \mid X_0 = i\right)$

$= E\left(\sum_{j=0}^{N} E(T+1 \mid X_0 = j)\, P_{ij}\right)$

$= 1 + p M_{i+1} + q M_{i-1}$ (1)

Let $W_i = M_i - M_{i-1}$

$\therefore 0 = 1 + p W_{i+1} + q(-W_i)$ (2)

$\therefore W_{i+1} = \frac{q}{p} W_i - \frac{1}{p}$

Also $M_i = W_1 + \cdots + W_N$

Verify by induction (student should do this)

$W_i = \left(\frac{q}{p}\right)^{i-1} W_1 - \frac{1}{p}\left(1 + \frac{q}{p} + \cdots + \left(\frac{q}{p}\right)^{i-2}\right)$

[1] $p = q = \frac{1}{2}$

$$W_i = W_1 - 2\sum_{j=0}^{i-2} 1 = W_1 - 2(i-1)$$

$$\therefore M_i = W_1 + \cdots + W_i = i W_1 - 2\sum_{j=0}^{i-1} j$$
$$= i W_1 - 2\,\frac{(i-1)\,i}{2} = i W_1 - i(i-1)$$

$$0 = W_1 + \cdots + W_N \quad (= M_N)$$
$$= N W_1 - N(N-1)$$
$$\therefore W_1 = (N-1)$$
$$\therefore M_1 = W_1 = (N-1)$$

and

$$M_i = i W_1 - i(i-1) = i(N-1) - i(i-1)$$
$$= i(N-1-i+1) = i(N-i)$$

(2) $p \neq \frac{1}{2}$

Here the direct method is cumbersome

Return to (1) and (2).
(2) is a non-homogeneous difference equation.
As with non-homogeneous DE's the general solution is of the form

(specific solution) + (general solution to homogeneous difference equation)

Specific solution
$m_i = c\,i$
Plug into (1) and (2)
Verify $c = \dfrac{1}{q-p}$, and $m_i = \dfrac{i}{q-p}$
solves (2).

For homogenous version of (2)
$0 = p\,W_{i+1} - q\,W_i$
$\therefore\ W_{i+1} = \dfrac{q}{p}\,W_i$

By iteration
$W_i = \left(\dfrac{q}{p}\right)^{i-1} W_1$

$$M_1' = W_1 \cdot \sum_1^{i} \left(\frac{q}{p}\right)^{j} = W_1 \cdot \sum_{j=0}^{i-1} \left(\frac{q}{p}\right)^{j}$$ (4)

$$= W_1 \frac{\left(1 - \left(\frac{q}{p}\right)^{i}\right)}{1 - \frac{q}{p}}$$

∴ general solution is

$$M_k = m_k + a \cdot M_k' + b$$

$$= \frac{k}{q-p} + a \cdot \frac{\left(1 - \left(\frac{q}{p}\right)^{k}\right)}{1 - \frac{q}{p}} + b \qquad (3)$$

a, b to be determined

$M_0 = 0$

∴ from (3)

$$0 = 0 + a \cdot \frac{(1-1)}{1 - \frac{q}{p}} + b = b \qquad (4)$$

$M_N = 0$

from (3) and (4)

$$0 = \frac{N}{q-p} + a \frac{\left(1 - \left(\frac{q}{p}\right)^{N}\right)}{1 - \frac{q}{p}}$$

$$= \frac{1}{q-p}\left(N - ap\left(1 - \left(\frac{q}{p}\right)^{N}\right)\right)$$

$$\therefore \quad a = \frac{N}{p} \cdot \frac{1}{\left(1 - \left(\frac{q}{p}\right)^{N}\right)}$$

$$\therefore\ M_k = \frac{k}{q-p} + a \cdot \frac{\left(1 - \left(\frac{q}{p}\right)^k\right)}{\left(1 - \frac{q}{p}\right)}$$ ⑤

$$= \frac{k}{q-p} + \frac{N}{p\left(1 - \left(\frac{q}{p}\right)^N\right)} \cdot \frac{\left(1 - \left(\frac{q}{p}\right)^k\right)}{1 - \frac{q}{p}}$$

$$= \frac{k}{q-p} - \frac{N}{q-p} \cdot \frac{\left(1 - \left(\frac{q}{p}\right)^k\right)}{\left(1 - \left(\frac{q}{p}\right)^N\right)}$$