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Appendix A.2 Countable Sets
W is a countable set if

$$\Omega = W = \{w_1, w_2, \dots\}$$
$$= \{w_n \mid n \geq 1\}$$
$$\exists f : \mathbb{N} \to W$$
$$\ni f(\mathbb{N}) = W = \Omega$$

Ω finite $\exists n \ni f(\{1, \dots, n\}) = \Omega$

$A \subseteq \Omega$, Ω is countable
then A is countable

$\mathbb{Z}$

$f(n)$

$f(1) = 0$
$f(2) = 1$
$f(3) = -1$
$f(4) = 2$
$f(5) = -2$
$\vdots$

$$\mathbb{N} \times \mathbb{N} = \{(n, m) \mid n \in \mathbb{N}, m \in \mathbb{N}\}$$

(1,1)	(2,1)	(3,1)	(4,1)	...
(1,2)	(2,2)	(3,2)		
(1,3)	(2,3)			
(1,4)	(2,4)			
(1,5)	(2,5)		(i,j)	

$$\begin{array}{cccc}
(1,1) & (2,1) & (3,1) & (4,1) \\
(1,2) & (2,2) & (3,2) & \\
(1,3) & (2,3) & & \\
(1,4) & (2,4) & & \\
(1,5) & (2,5) & & (i,j)
\end{array}$$

Ω_1, Ω_2 countable

$$\begin{array}{ccc}
f(1,1) & f(2,1) & f(3,1) \\
f(1,2) & f(2,2) & \\
f(1,3) & &
\end{array}$$

$\Omega_1 \times \Omega_2$ countable

$\mathbb{Z} \times \mathbb{Z}$ countable

Q = set of rationals

$\rightarrow \{ (n, m) \mid n, m \text{ mutally prime} \}$

$\frac{n}{m}$

$Q \subseteq \mathbb{Z} \times \mathbb{Z}$

$\therefore Q$ is countable

$[0,1]$ = reals from 0 to 1

?

Is $[0,1]$ countable?

Suppose it is countable

$$x = d_1 \cdot 10^{-1} + d_2 10^{-2} + d_3 10^{-3} + \cdots$$
$$= \sum_{i=1}^{\infty} d_i \cdot 10^{-i} \qquad d_i \in \{0, \ldots, 9\}$$

By assumption $[0,1] = \{x_1, x_2, \ldots\}$
ie countable $[0,1]$

$$x_1 = d_{11} \cdot 10^{-1} + d_{12} 10^{-2} + \cdots$$
$$x_2 = d_{21} \cdot 10^{-1} + d_{22} \cdot 10^{-2} + \cdots$$
$$\vdots$$
$$\rightarrow x_n = d_{n1} \cdot 10^{-1} + d_{n2} \cdot 10^{-2} + d_{n3} \cdot 10^{-3} + \cdots$$
$$\vdots$$

$$c = c_1 10^{-1} + c_2 10^{-2} + c_3 10^{-3} + \cdots$$
$$c_1 = d_{11} + 1 \pmod{10}$$
$$c_n = d_{nn} + 1 \pmod{10}$$

$\therefore$ this contradicts the assumption that $[0,1]$ is countable
$\therefore [0,1]$ is not countable.

Binary representation of $[0,1)$
$$\rightarrow x = \sum_{i=1}^{\infty} d_i \cdot 2^{-i}, \quad d_i \in \{0,1\}$$

Infinite sequence of coin tosses

Sample space?

$\Omega_1 = \{0,1\}$

$\Omega - \{0\ 1\} \times \{0\ 1\}$

$\Omega_1 = \{0,1\}$

$\Omega_2 = \{0,1\} \times \{0,1\}$

$\vdots$

$\Omega_n = \displaystyle\mathop{\times}_{i=1}^{n} \{0,1\}$

$\Omega_\infty = \displaystyle\mathop{\times}_{i=1}^{\infty} \{0,1\}$

$\rightarrow \quad = \{ \underset{\sim}{w} \mid \underset{\sim}{w} = (w_1, w_2, \dots)$
$w_i = 0 \text{ or } 1 \}$

Ω_∞

Return to Chapter 1

Is there a prob triple
$([0,1], \mathcal{F}, P)$

so this corresponds to Unif$([0,1])$
and for which

$\mathcal{F} = \text{set of all subsets of } [0,1]$

Main result:

$\mathcal{F} \neq \text{set of all subsets of } [0,1]$

"discrete" case

$\Omega = \{0\ 1\ 2 \quad \} = \mathbb{N}_0$

Discrete case

$\Omega = \{0, 1, 2, \dots\} = \mathbb{N}_0$

$\mathcal{F}$ = set of all subsets of Ω
= power set of Ω $= 2^{\Omega}$

$\alpha_k = P(\{\omega_k\}) = P(\{k\})$

$\alpha_k \geq 0, \quad \sum_{k=0}^{\infty} \alpha_k = 1$

$P(A) = P\left(\bigcup_{k \in A} \{k\}\right)$
$= \sum_{k \in A} P(\{k\}) = \sum_{k \in A} \alpha_k$

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back to $[0,1)$ Uniform example

$A \subseteq [0,1]$

$A \oplus r = \{ a + r \pmod 1 \mid a \in A \}$

$r = \frac{1}{2}$

$[0, \frac{1}{4}) \oplus r = [\frac{1}{2}, \frac{3}{4})$

$[\frac{7}{8}, 1) \oplus r = [\frac{3}{8}, \frac{1}{2})$

$(\frac{3}{8}, \frac{5}{8}) \oplus r = [\frac{7}{8}, 1) \cup (0, \frac{1}{8})$

Partition $[0,1)$

$x \sim y$ (x equivalent to y)
iff $x - y \in \mathbb{Q}$

H =: chose one element from each partition set.

$\mu(H \oplus r) = \mu(H)$

$\cdot \quad [0,1) = \bigcup_{r \in \mathbb{Q} \cap [0,1)} (H \oplus r)$

$f \qquad \mathbb{Q}(\,)$

$[0,1) = \bigcup_{r \in \mathbb{Q} \cap [0,1)} (H \oplus r)$

$(H \oplus r) \cap (H \oplus s) = \phi$ for $s, r \in \mathbb{Q} \cap [0,1)$

If $\mathcal{F}$ = set of all subsets of $[0,1)$

then $H \oplus r$ is measurable

$\therefore \; 1 = \mu([0,1))$

$= \mu\left(\bigcup_{r \in \mathbb{Q} \cap [0,1)} H \oplus r \right)$ ← disjoint union

$= \sum_{r \in \mathbb{Q} \cap [0,1)} \mu(H \oplus r)$

(countable additivity)

$= \sum_{r \in \mathbb{Q} \cap [0,1)} \mu(H)$

$\mu(H) = 0$ RHS $= 0$, LHS $= 1$ $\rightarrow\leftarrow$

$\mu(H) > 0$ RHS $= \infty$, LHS $= 1$ $\rightarrow\leftarrow$