

Appendix A.2 Countable Sets

W is a countable set if

$$\Omega \subseteq W = \{w_1, w_2, \dots\}$$

$$= \{w_n \mid n \geq 1\}$$

$$\exists f: \mathbb{N} \rightarrow W$$

$$\Rightarrow f(\mathbb{N}) = W = \Omega$$

$$\Omega \text{ finite } \exists n \text{ s.t. } f(\{1, \dots, n\}) = \Omega$$

$A \subseteq \Omega$, Ω is countable
then A is countable

 $\mathbb{Z}$

$$f(n)$$

$$f(1) = 0$$

$$f(2) = 1$$

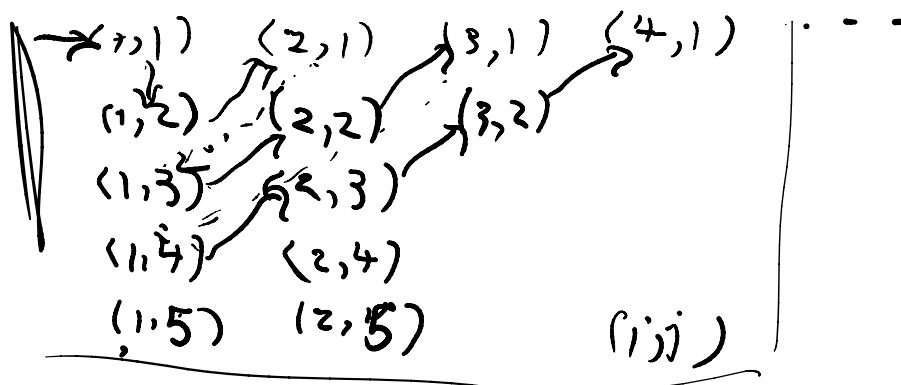
$$f(3) = -1$$

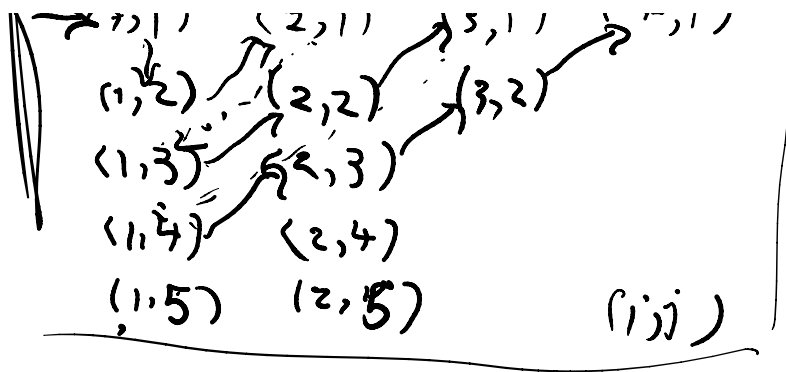
$$f(4) = 2$$

$$f(5) = -2$$

 $\vdots$

$$\mathbb{N} \times \mathbb{N} = \{(n, m) \mid n \in \mathbb{N}, m \in \mathbb{N}\}$$





Ω_1, Ω_2 countable

$f(1,1), f(2,1), f(3,1), \dots, f(1,2), f(2,2), f(1,3), f(2,3), \dots$ $\Omega_1 \times \Omega_2$ countable

$\mathbb{Z} \times \mathbb{Z}$ countable

$\mathbb{Q}$ = set of rationals

$\rightarrow \{(\underline{n}, m) \mid n, m \text{ mutually prime}\}$

$\frac{n}{m}$

$\mathbb{Q} \subseteq \mathbb{Z} \times \mathbb{Z}$

$\therefore \mathbb{Q}$ is countable

$[0, 1]$ = reals from 0 to 1

?

Is $[0,1]$ countable!

Suppose it is countable

$$\begin{aligned}x &= d_1 \cdot 10^{-1} + d_2 \cdot 10^{-2} + d_3 \cdot 10^{-3} + \dots \\&= \sum_{i=1}^{\infty} d_i \cdot 10^{-i} \quad d_i \in \{0, \dots, 9\}\end{aligned}$$

By assumption $[0,1] = \{x_1, x_2, \dots\}$
ie countable $[0,1]$

$$x_1 = d_{11} \cdot 10^{-1} + d_{12} \cdot 10^{-2} + \dots$$

$$x_2 = d_{21} \cdot 10^{-1} + d_{22} \cdot 10^{-2} + \dots$$

$$\rightarrow x_n = d_{n1} \cdot 10^{-1} + d_{n2} \cdot 10^{-2} + d_{n3} \cdot 10^{-3} + \dots$$

$$c = c_1 \cdot 10^{-1} + c_2 \cdot 10^{-2} + c_3 \cdot 10^{-3} + \dots$$

$$c_1 = d_{11} + 1 \pmod{10}$$

$$c_{11} = d_{n1} + 1 \pmod{10}$$

$\therefore$ this contradicts the assumption
that $[0,1]$ is countable

$\therefore [0,1]$ is not countable.

Binary representation of $[0,1]$

$$\rightarrow x = \sum_{i=1}^{\infty} d_i \cdot 2^{-i} \quad , d_i \in \{0,1\}$$

Infinite sequence of coin tosses

Sample space?

$$\Omega_1 = \{0,1\}$$

$$\Omega = \{0,1\} \times \{0,1\}$$

$$\Omega_1 = \{0,1\}$$

$$\Omega_2 = \{0,1\} \times \{0,1\}$$

$$\vdots$$

$$\Omega_n = \prod_{i=1}^n \{0,1\}$$

$$\Omega_\infty = \prod_{i=1}^{\infty} \{0,1\}$$

$$\rightarrow = \left\{ \underline{\omega} \mid \underline{\omega} = (\omega_1, \omega_2, \dots) \right. \\ \left. \omega_i = 0 \text{ or } 1 \right\}$$

$$\Omega_\infty$$

Return to Chapter 1

Is there a prob triple
 $(\mathcal{C}[0,1], \mathcal{F}, \mathbb{P})$

so this corresponds to $\text{Unif}([0,1])$
 and for which

$\mathcal{F} = \text{set of all subsets of } [0,1]$

Main result:

$\mathcal{F} \neq \text{set of all subsets of } [0,1]$

"discrete" case

$$\Omega = \{0,1,2\} \quad \mathcal{F} = \mathcal{P}_0$$

discrete case

$$\Omega = \{0, 1, 2, \dots\} = \mathbb{N}_0$$

$\mathcal{F}$ = set of all subsets of Ω
 = power set of $\Omega = 2^\Omega$

$$\alpha_k = P(\{\omega_k\}) = P(\{k\})$$

$$\alpha_k \geq 0, \quad \sum_{k=0}^{\infty} \alpha_k = 1$$

$$P(A) = P\left(\bigcup_{k \in A} \{k\}\right) \\ = \sum_{k \in A} P(\{k\}) = \sum_{k \in A} \alpha_k$$

back to $[0, 1)$ uniform example

$$A \subseteq [0, 1]$$

$$A \oplus r = \{a + r \pmod{1} \mid a \in \overset{A}{[0, 1]}\}$$

$$r = \frac{1}{2}$$

$$[0, \frac{1}{4}) \oplus r = [\frac{1}{2}, \frac{3}{4})$$

$$[\frac{7}{8}, 1) \oplus r = [\frac{3}{8}, \frac{1}{2})$$

$$(\frac{3}{8}, \frac{5}{8}) \oplus r = [\frac{7}{8}, 1) \cup [0, \frac{1}{8})$$

Partition $[0, 1)$

$$x \sim y \text{ (x equivalent to y)}$$

$$\text{iff } x - y \in \mathbb{Q}$$

H =: chose one element from each partition set.

$$\mu(H \oplus r) = \mu(H)$$

$$[0, 1) = \bigcup_{r \in \mathbb{Q} \cap [0, 1)} (H \oplus r)$$

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$\mathbb{Q} \cap [0, 1)$

$$\cdot \quad L^0([0,1]) = \bigcup_{r \in \mathbb{Q} \cap [0,1]} (H \oplus r)$$

$$(H \oplus r) \cap (H \oplus s) = \emptyset \text{ for } s, r \in \mathbb{Q} \cap [0,1]$$

If $\mathcal{F}$ = set of all subsets of $[0,1]$

then $H \oplus r$ is measurable

$$\therefore 1 = \mu([0,1])$$

$$= \mu\left(\bigcup_{r \in \mathbb{Q} \cap [0,1]} H \oplus r\right) \leftarrow \text{disjoint union}$$

$$= \sum_{\substack{r \in \\ \mathbb{Q} \cap [0,1]}} \mu(H \oplus r)$$

(countable additivity)

$$= \sum_{r \in \mathbb{Q} \cap [0,1]} \mu(H)$$

$$\mu(H) = 0 \quad \text{RHS} = 0, \text{ LHS} = 1 \rightarrow \leftarrow$$

$$\mu(H) > 0 \quad \text{RHS} = \infty, \text{ LHS} = 1 \rightarrow \leftarrow$$