January 18, 2016 2:30 PM

Sec 2.3 Extension Theorems

see Ex. 2.2.3 Semi algebra

Thm 2.3.1

(2.3.2)

$$P\left(\bigcup_{n=1}^{k} A_i\right) \geq \sum_{i=1}^{k} P(A_i) \text{ for } A_1,\dots,A_k \in \mathcal{J}$$

A_i's disjoint

(2.3.3) $A \subseteq \Omega$

$$P(A) \leq \sum_{n\geq 1} P(A_n)$$

$\forall A_1,\dots,A_n,\dots \ni A \subseteq \bigcup_{n\geq 1} A_i$

⋮

$\exists$ σ-algebra $\mathcal{M} \supseteq \mathcal{J}$ (such that)
and $\exists$ P^* defined on $\mathcal{M}$ $\ni$.
P^* is a prob. measure on $\mathcal{M}$

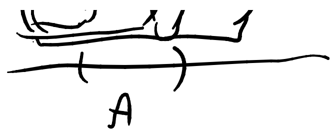
ie $(\Omega, \mathcal{M}, P^*)$ is a prob triple

Outer measure (2.3.4)

$$P^*(A) = \inf_{\substack{A_1, A_2 \dots \in \mathcal{J} \\ \ni A \subseteq \bigcup_{n\geq 1} A_n}} \sum_{i=1} P(A_i)$$

defined for all $A \subseteq \Omega$

A



Theorem 2.3.5

P^* is an extension of P on $\mathcal{J}$

ie if $A \in \mathcal{J}$ then $P^*(A) = P(A)$

Theorem 2.3.6

P^* is countably subadditive

Outline of the proof

B_n. for any $\epsilon \cdot 2^{-n} > 0$

$\exists\ C_{n1}, C_{n2}, C_{n3}, \dots \in \mathcal{J}$

$$P^*(B_n) \le \sum_{i=1} P(C_{ni}) \le P^*(B_1) + \epsilon \cdot 2^{-n}$$

$B_1, B_2, \dots \subseteq \Omega$

$$P^*\left(\bigcup_{n=1}^{\infty} B_n\right)$$

$$\le \sum_n \sum_i P(C_{ni}) \qquad \text{by def of } P^*, \inf$$

$$= \sum_{n=1}^{\infty} \left\{ \sum_{i=1}^{\infty} P(C_{ni}) \right\}$$

$$\le \sum_{n=1}^{\infty} \left\{ P^*(B_n) + \epsilon \cdot 2^{-n} \right\}$$

$$\le \sum_{n=1}^{\infty} P^*(B_n) + \sum_{n=1}^{\infty} \epsilon \cdot 2^{-n}$$

$$= \sum_{n=1}^{\infty} P^*(B_n) + \epsilon$$

$$\mathcal{M} = \left\{ A \subseteq \Omega \;\middle|\; P^*(E \cap A) + P^*(E \cap A^c) = P^*(E) \quad \forall \right\}$$

$= p^*(E)$

(2.3.7) $\forall E \subseteq \Omega \}$

Equivalent

$\mathcal{M} = \{ A \subseteq \Omega \mid p^*(E \cap A) + p^*(E \cap A^c) \leq p^*(E) \quad \forall E \subseteq \Omega \}$

(2.3.8)

Lemma 2.3.9

$A_1, A_2, \dots \in \mathcal{M}$, disjoint

$p^*(\bigcup_{n \geq 1} A_n) = \sum_{n=1}^{\infty} p^*(A_n)$

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pf: 2 sets

$p^*(A_1 \cup A_2) = p^*(\underline{A_1 \cap (A_1 \cup A_2)}) + p^*(A_1^c \cap (A_1 \cup A_2))$

$= p^*(A_1) + p^*(\underline{A_1^c \cap (A_1 \cup A_2)})$

$= p^*(A_1) + p^*(A_2)$

since $A_1^c \cap (A_1 \cup A_2) = A_2$

By induction, $A_1, \dots, A_m \in \mathcal{M}$, disjoint

$p^*(\bigcup_{n=1}^{m} A_n) = \sum_{n=1}^{m} p^*(A_n)$

$p^*(\bigcup_{n=1}^{m} A_n) \leq p^*(\bigcup_{n=1}^{\infty} A_n)$

$\sum_{n=1}^{m} p^*(A_n)$

$\therefore \sum_{n=1}^{\infty} p^*(A_n) \leq p^*(\bigcup_{n=1}^{\infty} A_n)$

Also by condition (2.3.3) & Lemma 2.3.6

$p^*(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} p^*(A_n)$

Show $\mathcal{M}$ is σ-algebra

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Lemma 2.3.10

$\mathcal{M}$ is an algebra

Pf.

$\emptyset, \Omega \in \mathcal{M}$

$A \in \mathcal{M}$

$P^*(A^c \cap E) + P^*((A^c)^c \cap E) = P^*(E)$

$\therefore A^c \in \mathcal{M}$

If A_1, A_2 show $A_1 \cap A_2 \in \mathcal{M}$

$$P^*((A_1 \cap A_2) \cap E) + P^*((A_1 \cap A_2)^c \cap E)$$
$$= P^*(A_1 \cap A_2 \cap E) + P^*(\underbrace{(A_1^c \cup A_2^c)} \cap E)$$
$$= P^*(A_1 \cap A_2 \cap E) + P^*((A_1^c \cap A_2 \cap E) \cup (A_1 \cap A_2^c \cap E) \cup (A_1^c \cap A_2^c \cap E))$$
$$\le P^*(A_1 \cap \underline{A_2 \cap E}) + P^*(A_1^c \cap \underline{A_2 \cap E}) + P^*(A_1 \cap A_2^c \cap E) + P^*(A_1^c \cap A_2^c \cap E)$$
$$= P^*(A_2 \cap E) + P^*(A_2^c \cap E)$$
$$= P^*(E)$$

Lemma 2.3.11 read this proof

Lemma 2.3.13

Lemma 2.~~3~~. 14

a b

Lemma 2.3.14

$\mathcal{M}$ is a σ-algebra

$A_1, A_2, \cdots \in \mathcal{M}$

disjoint

$D_1 = A_1$

$D_2 = A_2 \cap A_1^c = A_2 \setminus A_1 \qquad D_1 \cup D_2 = A_1 \cup A_2$

$D_3 = A_3 \cap (A_1 \cup A_2)^c = A_3 \setminus (A_1 \cup A_2)$

$\vdots$

$D_i = A_i \cap (A_1 \cup \cdots \cup A_{i-1})^c = A_i \setminus (A_1 \cup \cdots \cup A_{i-1})$

$\vdots$

Lemma 2.3.15

$\mathcal{J} \subseteq \mathcal{M}$

see also p15 top

If $A \in \mathcal{M}$, $B \subseteq A$

and $P^*(A) = 0$

Is $B \in \mathcal{M}$?

Sec 2.4

$\Omega = [0, 1]$

$\mathcal{J} = \{$ all intervals of $[0,1]$ $\}$

P on $\mathcal{J}$ by $(a,b) \in \mathcal{J}$

$P((a,b)) = b - a$

2

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$\mathbb{R}^2$ $\quad [0,1) \times [0,1)$