January 25, 2016 2:38 PM

Lebesgue measure

→ $\mathcal{J}$ = semialgebra

$\beta(\mathcal{J})$ = smallest σ-algebra containing $\mathcal{J}$

$\equiv$ Borel σ-field containing $\mathcal{J}$

$\mathcal{F}_\alpha$ σ-fields $\alpha \in I$

$$\mathcal{G} = \bigcap_{\alpha \in I} \mathcal{F}_\alpha$$

$$\beta(\mathcal{J}) = \bigcap_{\alpha} \mathcal{F}_\alpha$$

$\mathcal{F}_\alpha$ = σ-field

$\alpha \in$ Index set

$\mathcal{F}_\alpha$ = σ-field containing $\mathcal{J}$

Exercise 2.4.5

$A = \{(-\infty, x] \mid x \in R\}$

→ $\sigma(A)$ = smallest σ-field containing A

$A \in \mathcal{A}$ $\therefore (x, \infty) \in \sigma(A)$

→ $(-\infty, x) = \bigcup_{n=1}^{\infty} (-\infty, x - \frac{1}{n}] \in \sigma(A)$

→ $(-\infty, x] = \bigcap_{n=1}^{\infty} (-\infty, x + \frac{1}{n}]$

$(a, b]$ can we write this as an element of $\sigma(A)$?

this as an

$\sigma(A)$?

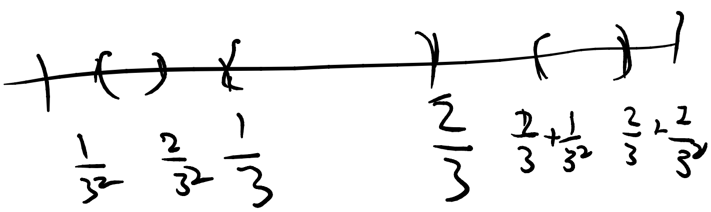
$(-\infty, x]^c = (x, \infty) \in \sigma(A)$

$a<b$ $(a, \infty) \cap (-\infty, b] = (a, b] \in \sigma(A)$

$[a, b] = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b]$

$Q((-\infty, x])$

Cantor Set = K



$K \overset{?}{\in} \mathcal{B}$

or equivalently K is a countable union of intervals ($K \in \mathcal{B}_1$, p 20 (2.2.6))

elements of K

$x = \sum_{n=1}^{\infty} c_n \cdot 3^{-n}$, $c_n \in \{0, 2\}$

Show $K \notin \mathcal{B}_1$.

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§2.5

Corollary 2.5.1 $P(\emptyset) = 0$, $P(\Omega) = 1$

$\supset$ $\mathcal{J}$ [1) $\mathcal{J}$-semialgebra

Corollary 2.5.1 $P(\phi)=0$, $P(\Omega)=1$

$P: \mathcal{J} \to [0,1]$, $\mathcal{J}$ = semialgebra

satisfies (2.3.2)

and countable subadditivity

$$P\left(\bigcup_{n\geq} B_n\right) \leq \sum_{n\geq 1} P(B_n)$$

$\forall\ B_j \in \mathcal{J}$, with $\bigcup_{n\geq 1} B_n \in \mathcal{J}$

Then $\exists\ P^*, \mathcal{M}$ s.t. $(\Omega, \mathcal{M}, P^*)$

prob triple and $P^*(A) = P(A)$

$\forall A \in \mathcal{J}$

Read proof

Corollary 2.5.4

$\mathcal{J}$ semialgebra

$P: \mathcal{J} \to [0,1]$

$P(\phi)=0$, $P(\Omega)=1$

satisfying

$$P\left(\bigcup_{n=1}^{\infty} D_n\right) = \sum_{n=1}^{\infty} P(D_n) \qquad (2.5.5)$$

for all disjoint $D_n \in \mathcal{J}$, s.t.

$\bigcup_{n\geq 1}^{\infty} D_n \in \mathcal{J}$

Then $\exists$ prob triple $(\Omega, \mathcal{M}, P^*)$

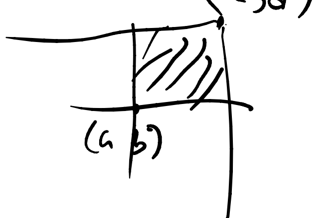
such that $P^*(A) = P(A)\ \ \forall A \in \mathcal{J}$

$$F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(u,v)\,du\,dv$$

F 2-d cdf

$P((a,b] \times (c,d])$

$= F(c,d)$

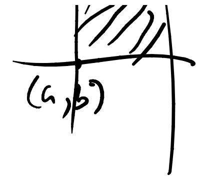


$= F(c,d)$

$- F(a,d)$

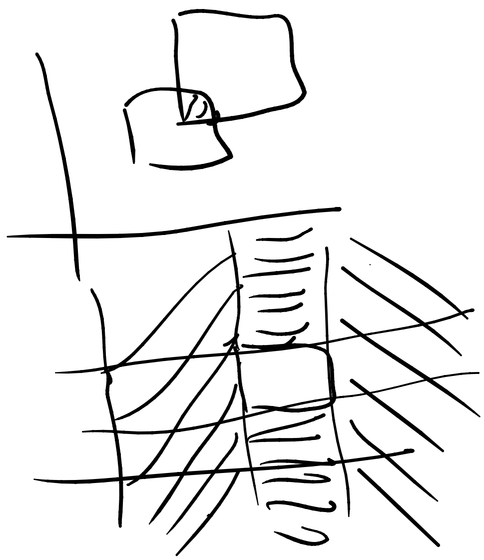
$- F(c,d)$

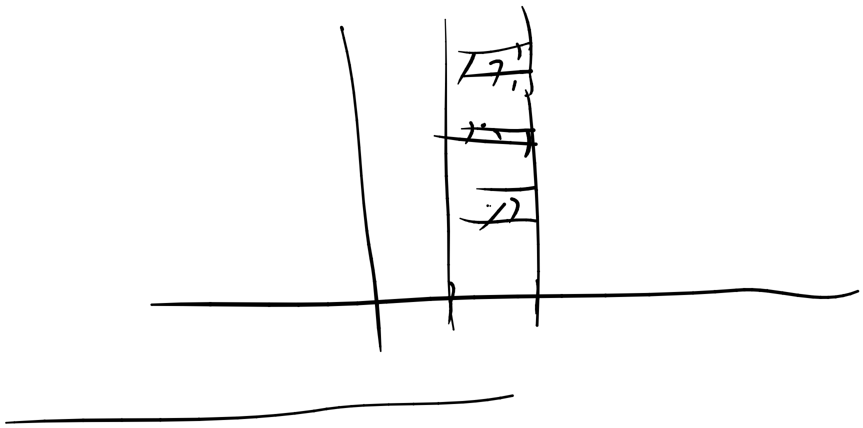
$+ F(a,b)$



$\mathcal{J}$ = cartesian "rectangles"

ie intervals on one axis × intervals on the other axis





Proof of Corollary 2.5.4

Show (2.3.3)

ie show

$A_1, A_2, \dots \in \mathcal{J}$, $A \subseteq \bigcup_{j\ge1} A_j$

$P(A) \le \sum_{n\ge1} P(A_n)$

$A \subseteq B \qquad P(A) \le P(B)$

$B = A \cup A^c$

$A^c = J_1 \cup \dots \cup J_k$, J_l's disjoint

$$
\begin{aligned}
P(B) &= P(A) + P(A^c) \\
&\ge P(A) + P(B \cap A^c) \\
&= P(A) + \sum P(\underbrace{B \cap J_k}) \\
&\ge P(A)
\end{aligned}
$$

$B_1, B_2, \dots, \quad A = \bigcup_{j=1} B_j \in \mathcal{J}$

$D_1 = B_1$

$D_2 = B_2 \cap B_1^c$

$D_3 = B_3 \cap (D_1 \cup D_2)^c$

$\vdots$

D's disjoint

$\bigcup_j B_j = \bigcup_j D_j$

$$
\begin{aligned}
P\Big(\bigcup_n B_n\Big) &= P\Big(\bigcup_n D_n\Big) \\
&= P\Big(\bigcup_n \bigcup_{j=1}^{k_n} J_{n,j}\Big) \\
&= \sum_n \underbrace{\sum_{j=1}^{k_n} P(J_{n,j})} \qquad (2.5.5) \\
&= \sum (\qquad)
\end{aligned}
$$

$$\cancel{= \{ P(B_n) \}}$$

$$B_n = \bigcup_{m \le n}^{n} \bigcup_{j=1}^{k_m} \left(J_{m,j} \cap B_n \right)$$

for given m disjoint

I will return to this next day.

Prop. 2.5.7

uniqueness.

$\mathcal{J}$ semialgebra

P defined on $\mathcal{J}$ satisfying Theorem 2.3.1 or one of Corollaries

$(\Omega, \mathcal{M}, P^*)$ ~~ets~~ probtriple from extension.

Q another prob. measure on $\mathcal{F}$, σ-algebra, $\mathcal{J} \subseteq \mathcal{F} \subseteq \mathcal{M}$ and $Q(A) = P(A)$ for all $A \in \mathcal{J}$

Then $Q = P^*$ on $\mathcal{F}$

$$P^*(A) = \inf_{\substack{A_1, A_2, \dots \in \mathcal{J} \\ A \subseteq \bigcup_j A_j}} \sum_j P(A_j)$$

$A_1, A_2, \dots$

$A \subseteq \bigcup_j A_j$

$$= \inf_{\substack{A_1, \dots \in \mathcal{J} \\ A \subseteq \bigcup_j A_j}} \underbrace{\sum_j Q(A_j)}_{\geq \text{ by } (2.3.3)}$$

$$\geq \inf_{\substack{A_1, \dots \\ A \subseteq \bigcup_j A_j}} Q\left(\bigcup_j A_j\right)$$

$$\geq \quad '' \quad Q(A)$$

$$= Q(A)$$