February 1, 2016 2:33 PM

Chapter 3 : r.v.s

- X is a r.v. If it is measurable

$(\Omega, \mathcal{F}, P)$ prob triple.

$X : \Omega \to \mathbb{R}$ (can also have $X : \Omega \to \Omega'$ $(\Omega', ?, P')$ prob triple

$\mathcal{B}(\mathbb{R})$ Borel sets

$\forall A \in \mathcal{B}(\mathbb{R})$,

$X^{-1}(A) = \{\omega \in \Omega \mid X(\omega) \in A\}$

$X^{-1}((-\infty, a])$, $X^{-1}((a, b])$

X is said to be measurable if
$X^{-1}(A) \in \mathcal{F}$ for all $A \in \mathcal{B}(\mathbb{R})$

simpler version
X is measurable if
$X^{-1}(I) \in \mathcal{F}$ for all intervals I

$B = X^{-1}(I)$
then $P(B)$ is defined
and so $P(X \in I)$
$= P(\{\omega \mid X(\omega) \in I\})$
$= P(X^{-1}(I))$

Eg: X_1, X_2, X_3 "iid" r.v.
taking values $-1, 1$
$\Omega = \{(\omega_1, \omega_2, \omega_3) \mid \omega_i = 0 \text{ or } 1\}$
$P(\{\omega\}) = \frac{1}{8}$ $\mathcal{F}$ = powerset

$\Omega = \{(w_1, w_2, w_3) \mid w_i \ldots\}$

$P(\{w\}) = \frac{1}{8}$ $\quad \mathcal{F} =$ powerset on Ω

$Y_1 = X_1$

$Y_2 = X_1 + X_2$

$X_1(w) = \begin{cases} 1 & \text{if } w_1 = 1 \\ -1 & \text{if } w_1 = 0 \end{cases}$

$X_2(w) = \begin{cases} +1 & \text{if } w_2 = 1 \\ -1 & \text{if } w_2 = 0 \end{cases}$

$X_3(w) = \begin{cases} +1 & \text{if } w_3 = 1 \\ -1 & w_3 = 0 \end{cases}$

$X_1^{-1}(\{1\}) = \{(1,0,0), (1,0,1), (1,1,0), (1,1,1)\}$
$\in \mathcal{F}$

$\mathcal{F}_1 = \sigma(X) =$ smallest σ-field so that X is r.v. (ie measurable w.r.t. $\mathcal{F}_1$)

$= \{ \{(1, w_2, w_3)\}_{w_2, w_3 = 0,1}, \{(0, w_2, w_3) \mid w_2, w_3 = 0 \text{ or } 1\}, \phi, \Omega \}$

$X_2^{-1}(\{1\}) \notin \mathcal{F}_1$

X_1 measurable w.r.t $\mathcal{F}_1$

$Y_2(w) = (X_1 + X_2)(w) = X_1(w) + X_2(w)$

$= \begin{cases} -2 & \text{if } X_1 = x_1, X_2 = x_2, \; x_1 + x_2 = -1 \\ \cancel{-1} & \\ 0 & \\ 1 & \\ 2 & \end{cases}$

$= \begin{cases} -2 & \text{if } w \in \{(0,0,0), (0,0,1)\} \\ \cancel{-1} & \\ 0 & \text{if } w \in \{(0,1,0), (0,1,1), (1,0,0), (1,1,0)\} \\ 2 & \text{if } w \in \{(1,1,0), (1,1,1)\} \end{cases}$

$\mathcal{F} =$ smallest σ-algebra

$\mathcal{F}_2$ = smallest σ-algebra containing ~~and~~ $\mathcal{F}_1$

$(\Omega, \mathcal{F}_1, P)$ $Y_1 = X_1$ measurable

Y_2 not measurable w.r.t $\mathcal{F}_1$

$(\Omega, \mathcal{F}_2, P)$

$Y_1^{-1}(A)\ X_1^{-1}(\{1\}) = \{(1,0,0), (1,0,1), (1,1,0), (1,1,1)\}$

Y_2 is measurable w.r.t $\mathcal{F}_2$

Prop 3.1.5 $(\Omega, \mathcal{F}, P)$

(a) $A \in \mathcal{F}$, I_A r.v.

(b) X, Y r.v., $X+c, cX, X^2, X+Y$ are r.v.

(c) $\{Z_n\}_{n\geq 1}$ sequence of r.v's

Suppose $\lim_{n\to\infty} Z_n(w) = Z(w)$ $\Leftarrow$

then Z is a r.v

$\{w \mid Z(w) \leq x\} = \{w \mid Z(w) \in (-\infty, x]\} \overset{?}{\in} \mathcal{F}$

If $a_n \to a$

then $\forall \epsilon > 0$, $\exists\ n_0(\epsilon) \ni$

$|a_n - a| \leq \epsilon \quad \forall n \geq n_0(\epsilon)$

$\therefore$ if $\epsilon > 0$, and if $a \leq x$

then $\exists\ n_0 \ni$. $a_n \leq x + \epsilon \quad \forall n \geq n_0$

$\bigcap^{\infty} \bigcup^{\infty} \bigcap^{\infty} \{w \mid Z_k(w) \leq x + \frac{1}{m}\}$ $\leftarrow$

$$\bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \left\{ \omega \mid Z_k(\omega) \le x + \frac{1}{m} \right\} \leftarrow$$

$$= \{ \omega \mid Z(\omega) \le x \}$$

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do Exercise 3.1.7

If f is Borel-measurable

$f : \mathbb{R} \to \mathbb{R}$

and if X is a r.v. on $(\Omega, \mathcal{F}, \cdot)$

then $f(X)$ is a r.v.

$Z = f(X) \qquad Z : \Omega \to \mathbb{R}$

$\Omega \xrightarrow{X} \mathbb{R} \xrightarrow{f} \mathbb{R}$

f Borel measurable

$f^{-1}(A) \in \mathcal{B} \quad \forall A \in \mathcal{B}$

$Z^{-1}(B) \qquad B \in \mathcal{B}$

$= \{ \omega \mid f(X(\omega)) \in B \}$

$= \{ \omega \mid X(\omega) \in \underbrace{f^{-1}(B)}_{\in \mathcal{B}} \}$

$= \underbrace{X^{-1}(\underbrace{f^{-1}(B)}_{\in \mathcal{B}})}_{\in \mathcal{B}}$ ← since f is measurable; ← since X is r.v.

$$(X, Y)^{-1}(B) = \{ \omega \mid (X(\omega), Y(\omega)) \in B \}$$

Remark 3.1.4

$(\Omega, \mathcal{F}, P)$ complete

X r.v.

$\Rightarrow \{\omega \mid X(\omega) \neq Y(\omega)\} \subseteq \Omega$

$P(\{\omega \mid X(\omega) \neq Y(\omega)\}) = 0$

$\Rightarrow Y$ is a r.v.

$\{X_\alpha \mid \alpha \in I\}$ family of r.v.'s

It is said to be a family of independent r.v.'s iff $\{X_{\alpha_1}, \dots, X_{\alpha_n}\}$ is a finite collection of independent r.v.'s, for all distinct $\alpha_1, \dots, \alpha_n \in I$

Example of n events such that any $n-1$ of these are independent but all n are (jointly) dependent.

Prototype example. built up from 2 fair coin tosses;

$\underbrace{Y_1, Y_2}_{\text{iid coin fair}}, \; Y_3 = Y_1 + Y_2 \pmod 2$

$A_1 = \{Y_1 = 1\}$

$A_2 = \{Y_2 = 1\}$

$\{Y_3 = 1\}$

Move Monday times to 2-4 EST (11AM-1PM) PST

affect Chinese Lunar New Year 2016 (Feb 8, 2016)

$A_2 = \{Y_2 = 1\}$

$A_3 = \{Y_3 = 1\}$

2016 (Feb 8, 2016)

$\Omega = \{ (0,0), (0,1), (1,0), (1,1) \}$

$P: \quad \frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4}$

Any 2 of A_1, A_2, A_3 are independent

$A_1 \cap A_3 = \{(1,0),(1,1)\} \cap \{(0,1),(1,0)\}$

$= \{(1,0)\}$

$P(A_1 \cap A_3) = P(A_1) \cdot P(A_3) = \frac{1}{4}$ ✓

→ check also A_1, A_2 and A_2, A_3

$A_1 \cap A_2 \cap A_3 = \phi$

$\therefore P(A_1 \cap A_2 \cap A_3) = 0 \neq \frac{1}{8} = \prod_{i=1}^{3} P(A_i)$