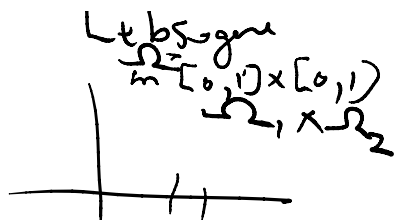


$$f: \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$\mathcal{B}(\mathbb{R}^2) \quad \mathcal{B}(\mathbb{R})$$

f is Borel measurable iff

$$\forall B \in \mathcal{B}(\mathbb{R})$$

$$f^{-1}(B) \in \mathcal{B}(\mathbb{R}^2)$$


$$X^*(\omega_1, \omega_2) = X(\omega)$$

$$Y^*(\omega_1, \omega_2) = Y(\omega)$$

Prop 3.2.3

X, Y independent r.v.'s $(X, Y: \Omega \rightarrow \mathbb{R})$

f, g measurable functions $(X, Y): \Omega \rightarrow \mathbb{R}^2$

$\Rightarrow f(X), g(Y)$ r.v.'s

then $f(X), g(Y)$ are independent r.v.'s

Pf:

$$A, B \in \mathcal{B}(\mathbb{R})$$

$$P(f(X) \in A, g(Y) \in B)$$

$$= P(X \in f^{-1}(A), Y \in g^{-1}(B))$$

$$= P(\{\omega \mid X(\omega) \in f^{-1}(A)\} \cap \{\omega \mid Y(\omega) \in g^{-1}(B)\})$$

Prop 3.2.4

X, Y defined on $(\Omega, \mathcal{F}, P)$

X, Y independent iff $\forall x, y \in \mathbb{R}$

$$P(X \leq x, Y \leq y)$$

$$= P(X \in (-\infty, x], Y \in (-\infty, y])$$

$$= P(X \in (-\infty, x]) \cdot P(Y \in (-\infty, y])$$

3.3 Continuity of Probabilities

Prop 3.31

→ $A_n \uparrow A$

Then

$$P(A_n) \uparrow P(A)$$

Pf:

$$A_n \subseteq A_{n+1}$$

$$B_1 = A, \quad B_n = A_n \cap A_{n+1}^c \in \mathcal{F}$$

$$A = \bigcup_{n=1}^{\infty} B_n \quad \left(= \bigcup_{n=1}^{\infty} A_n \right)$$

$$P(A) = \sum_{n=1}^{\infty} P(B_n)$$

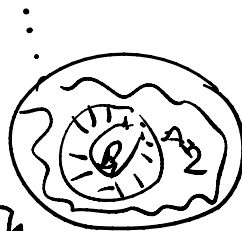
$$= \lim_{m \rightarrow \infty} \left(\sum_{n=1}^m P(B_n) \right)$$

$$= \lim_{m \rightarrow \infty} P\left(\bigcup_{n=1}^m B_n\right)$$

$$= \lim_{m \rightarrow \infty} P(A_m)$$

$$\text{Also } P(A_m) \leq P(A_{m+1})$$

$$\therefore P(A_m) \uparrow P(A)$$



§3.4

$$A_1, A_2, \dots \in \mathcal{F}$$

$$\{A_n \text{ infinitely often}\} = \{A_n \text{ i.o.}\}$$

$$= \left\{ \omega \mid \omega \in A_n \text{ for infinitely many } n \right\}$$

$$= \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

$$= \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

$$\{A_n \text{ finitely often}\}^c$$

$$\{A_n \text{ almost always}\}$$

$$= \{A_n \text{ a.a.}\}$$

$$= \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

Exercise
 3.6.3 (b) Can ~~anyone~~^{you} give an
 example.

3.6.4