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SLLN 2nd version

WLOG $X_i \geq 0$ $\qquad Y_i = X_i I(X_i \leq i)$

$S_n' = \sum_{i=1}^n Y_i \qquad P(X_i \neq Y_i \text{ i.o.}) = 0$

Need to show $\frac{1}{n} S_n' \to m = E(X_1)$

$E\left(\frac{1}{n}\sum_{i=1}^n Y_i\right) = \frac{1}{n}\sum_{i=1}^n E(X_i I(X_i \leq i)) = \frac{1}{n}\sum_{i=1}^n E(X_1 I(X_1 \leq i))$

$E(X_1 I(X_1 \leq i)) \to E(X_1) = m$ as $i \to \infty$

Show $\frac{1}{n}\sum_{i=1}^n E(X_1 I(X_1 \leq i)) \to m$

do this by showing: if $a_i \to a \in \mathbb{R}$ then $\frac{1}{n}\sum_{i=1}^n a_i \to a$

Fix $\alpha > 1$ $\qquad [y]$ = greatest integer $\leq y$

$u_m = [\alpha^m] \nearrow \infty$

Fix $x > 0$

$$\to \sum_{u_n \geq x} \frac{1}{u_n} \leq \sum_{\alpha^n \geq x} \frac{1}{u_n} = \sum_{k = \log_\alpha(x)} \frac{2}{[\alpha^k]}$$

$$\leq \sum_{k=\log_\alpha(x)}^{\infty} \frac{2}{\alpha^k}$$

$$= \frac{2 \cdot \alpha^{-\log_\alpha(x)}}{1 - \frac{1}{\alpha}} = \frac{\frac{2}{x}}{1 - \frac{1}{\alpha}}$$

Consider

$$\frac{S_{u_n}' - E(S_{u_n}')}{u_n} \quad \left(\text{show} \xrightarrow{a.s.} 0 \text{ on this subsequence } u_n \to \infty\right).$$

$$\sum_{n=1}^\infty P\left(\left|\frac{S_{u_n}' - E(S_{u_n}')}{u_n}\right| \geq \epsilon\right)$$

$$\leq \sum_{n=1}^\infty \frac{\operatorname{Var}\left(\frac{S_{u_n}'}{u_n}\right)}{\epsilon^2}$$

$$\leq \sum_{n=1}^\infty \frac{1}{\epsilon^2 u_n^2}\left(E((S_{u_n}')^2) - (E(S_{u_n}'))^2\right)$$

$$\leq \sum_{n=1}^\infty \frac{1}{\epsilon^2 u_n^2} E((S_{u_n}')^2)$$

$$= \sum_{n=1}^\infty \frac{1}{\epsilon^2 u_n^2}\left\{E(X_1^2 I(X_1 \leq 1)) + \cdots + E(X_{u_n}^2 I(X_{u_n} \leq u_n))\right\}$$

$$\left(\cancel{\leq \sum_{n=1}^\infty \frac{1}{\epsilon u_n^2} \cdot u_n E(X_1^2 I(X_1 \leq u_n))}\right.$$

$$\leq \frac{1}{\epsilon^2}\sum_{n=1}^\infty \frac{u_n}{u_n^2} E(X_1^2 \cdot I(X_1 \leq u_n))$$

$$= \frac{1}{\epsilon^2} E\left(X_1^2 \underbrace{\sum_{n=1}^\infty \frac{1}{u_n} I(u_n \geq X_1)}_{\sum_{u_n \geq x} \frac{1}{u_n}}\right)$$

$$\left(\quad {}^2 \quad \frac{2}{} \quad\right)$$

ϵ^2 $\sum_{u_n \geq X_1} \frac{1}{u_n}$

$$\leq \frac{1}{\epsilon^2} E\left(X_1^2 \cdot \frac{\frac{2}{X_1}}{1-\frac{1}{\alpha}} \right)$$

$$= \frac{2}{\epsilon^2(1-\frac{1}{\alpha})} E\left(\frac{X_1^2}{X_1} \right) = \frac{2}{\epsilon^2(1-\frac{1}{\alpha})} \cdot E(X_1) < \infty$$

$\therefore$ by Borel-Cantelli and Lemma 5.2.1

$$\frac{S'_{u_n} - E(S'_{u_n})}{u_n} \to 0 \text{ a.s.}$$

since $\frac{1}{u_n} \cdot E(S'_{u_n}) \to m$ as $n \to \infty$, then for any subsequence $u_n \to \infty$

(ie if $b_n \to b$, $b_{u_n} \to b$)

$$\therefore \frac{S'_{u_n}}{u_n} \to m \text{ a.s.}$$

Last step: Show $\frac{1}{n} S'_n \to m$ ie for all integers $n \geq 1$ not just this special subsequence u_n

$\exists A_\alpha \in \mathcal{F}$, $P(A_\alpha) = 1$

$$\forall w \in A_\alpha \quad \frac{S'_{u_n}(w) - E(S'_{u_n})}{u_n} \to m$$

$w \in A_\alpha$

$$? \leq \frac{S'_n(w)}{n} \leq ?$$

For any n, $\exists m$ s.t. $u_m \leq n < u_{m+1}$

$$\frac{u_m}{n} \cdot \frac{S'_{u_m}(w)}{u_m} \leq \frac{S'_{u_m}(w)}{n} \leq \frac{S'_n(w)}{n} \leq \frac{S'_{u_{m+1}}(w)}{n} \leq \underbrace{\frac{S'_{u_{m+1}}(w)}{u_{m+1}}}_{\to m} \cdot \frac{u_{m+1}}{n}$$

$\frac{u_m}{n} \cdot \frac{S'_{u_m}(w)}{u_m} \to m$

$$\leq \frac{S'_{u_{m+1}}}{u_{m+1}} \cdot \frac{u_{m+1}}{u_m}$$

$\frac{\alpha^{m+1}}{\alpha^m} = \alpha$

$$\frac{u_m}{u_{m+1}} \cdot \frac{S'_{u_m}(w)}{u_m} \leq$$

$$\frac{1}{\alpha} \cdot \frac{S'_{u_n}(w)}{u_n}$$

$\therefore$ if $w \in A_\alpha$,

$$\frac{1}{\alpha} \cdot m = \liminf_{n\to\infty} \frac{S'_{u_n}(w)}{u_n}$$

$$\leq \liminf_{n\to\infty} \frac{S'_n(w)}{n}$$

$$\leq \limsup_{n\to\infty} \frac{S'_n(w)}{n}$$

$$\leq \limsup_{n\to\infty} \frac{S'_{u_{m+1}}(w)}{u_{m+1}} \cdot \alpha$$

$$= m \cdot \alpha$$

Since $\alpha > 1$ is arbitrary, use $\alpha = 1 + \frac{1}{k}$

$\therefore$ if $w \in A_{(1+\frac{1}{k})}$, above holds.

Let $A = \bigcap_{k=1}^{\infty} A_{\frac{1}{k}}$.

$$P(A^c) = P\left(\bigcup (A_{\frac{1}{k}})^c\right) \leq \sum P(A^c_{\frac{1}{k}}) = 0$$

Let $A = \bigcap_{k=1} A_k$
$P(A^c) = P(\bigcup_k (A_k)^c) \le \sum P(A_k^c)$
$\therefore P(A) = 1$
$\forall \omega \in A$, then $\forall k \ge 1$

$$\underline{\frac{1}{1+\frac{1}{k}} \cdot m} \le \underline{\liminf_{n\to\infty} \frac{S_n'(\omega)}{n}} \le \limsup \frac{S_n'(\omega)}{n} \le m(1+\frac{1}{k})$$

$$\therefore \lim_{k\to\infty} \frac{1}{1+\frac{1}{k}} \cdot m \le \liminf \frac{S_n'(\omega)}{n} \le \limsup \frac{S_n'(\omega)}{n} \le \lim_{k\to\infty} m \cdot (1+\frac{1}{k}) = m$$

$$\therefore \lim_{n\to\infty} \frac{S_n'(\omega)}{n} = m$$

Chapter 6

Distributions of r.v's and Change of Variables Th'm.

X r.v. ie $\exists (\Omega, \mathcal{F}, P)$ prob triple such that X is measurable.

Then $\forall B \in$ Borel sets $= \mathcal{B}$, define

$\mu(B) = P(X \in B) = P \circ X^{-1}(B)$

μ is a measure on $\mathcal{B}$, $\mu(\mathbb{R}) = 1$,

so $(\mathbb{R}, \mathcal{B}, \mu)$ is also a prob. triple

μ is the "distribution" of X in the sense, we can define $X' : \mathbb{R} \to \mathbb{R}$, $X'(y) = y$ $\forall y \in \mathbb{R}$

X' is a r.v. on $(\mathbb{R}, \mathcal{B}, \mu)$, $\mu(X' \in B) = P(X \in B)$

Change of Variable Theorem

Theorem 6.1.1

Given a prob triple $(\Omega, \mathcal{F}, P)$, let X be r.v. with distribution μ. Then for any Borel measurable f,

$$E_P(f(X)) = \int_\Omega f(X(\omega))\, P(d\omega)$$
$$= \int_{\mathbb{R}} f(x)\, \mu(dx)$$

Proof.

① $B \in \mathcal{B}$ $\quad f = I_B$

$$\int_\Omega f(X(\omega)) P(d\omega) = E_P(f(X))$$
$$= \int_\Omega I_B(X(\omega)) P(d\omega)$$
$$= P(X \in B)$$
$$= \mu(B)$$
$$= \int I_B(x)\, \mu(dx)$$

② B_1, B_2 disjoint $\in \mathcal{B}$ $\quad f(x) = a_1 I_{B_1}(x) + a_2 I_{B_2}(x)$

verify LHS = RHS

③ Verify LHS = RHS for any simple f

④ $f \ge 0$. Use $f_n \uparrow f$, apply Monotone convergence theorem

⑤ $f = f^+ - f^-$, $f^+ \ge 0$, $f^- \ge 0$

Read Corollary 6.1.3

Law(X) = Law(Y) (even if defined on different ob. triples) iff $(f(Y))$ for all f measurable

$\mathcal{L}aw(X) = \mathcal{L}aw(Y)$ (on
prob. triples) iff
$E(f(X)) = E(f(Y))$ for all f measurable

$E_{\Omega_1}(f(X))$ $E_{\Omega_2}(f(Y))$

$E_\mu(f(X)) = E_\mu(f(Y))$
(on $\mathbb{R}$)

Corollary 6.1.4
If $P(X=Y)=1$ then $E(f(X)) = E(f(Y))$

Examples §6.2
① in some cases μ has a density, say ϕ
$\mu(B) = \int_B \phi(x)\,dx$ (Lebesgue integral on $\mathbb{R}$)
$= \int_B \phi(x)\,\lambda(dx)$ λ = Lebesgue measure

Then
$E(f(X)) = \int_{\mathbb{R}} f(x)\,\mu(dx)$
$= \int_{\mathbb{R}} f(x)\cdot\phi(x)\,dx$

Eg:
$\mathcal{L}aw(X) = \frac{1}{4}\delta_1 + \frac{1}{4}\delta_2 + \frac{1}{2}\mu_N$

δ_1 = prob. measure all mass at 1
δ_2 = " " " " " 2
μ_N = prob. measure of standard Normal
$E(f(X)) = \int_{\mathbb{R}} f(x)\,\mu(dx)$ μ = measure of law of X = distribution of X

$a = E(X) = \int_{\mathbb{R}} x\,\mu(dx)$
$= \frac{1}{4}\int_{\mathbb{R}} x\cdot\delta_1(dx) + \frac{1}{4}\int_{\mathbb{R}} x\,\delta_2(dx)$
$+ \frac{1}{2}\int_{\mathbb{R}} x\,\phi(x)\,dx$ $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$

$E(X^2) = ?$

$E((X-a)^2) = ?$

Exercise 6.3.1

Chapter 7
§7.1
Theorem 7.1.1
Let $\mu_1, \mu_2, \dots$ be a sequence of prob measures on $\mathbb{R}$ (Borel)
then $\exists$ r.v's $X_1, X_2, \dots$ and prob triple $(\Omega, \mathcal{F}, P)$ ($\ni P$) $\mathcal{L}aw(X_i) = \mu_i$ and

Let $\mu_1, \mu_2, \ldots$ and prob triple
Then $\exists$ r.v.'s $X_1, X_2, \ldots$ and
$\ni$ X_i r.v. on $(\Omega, \mathcal{F}, P)$, $\mathcal{L}aw(X_i) = \mu_i$ and
X_i's are independent.

Lemma 7.1.2
U is Uniform $(0,1)$ r.v. on prob triple $([0,1], \mathcal{B}, \lambda)$
F = ~~cdf~~ cumulative distribution function
i.e. $P(X \le x) = F(x)$, F is cdf of Law of X)
Set $\phi(u) = \inf(x \mid F(x) \ge u)$, $0 < u < 1$
Then $P(\phi(U) \le x) = F(x)$

Aside: $\phi(U)$ has distribution F