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Chap 7

Theorem 7.1.1

$\mu_1, \mu_2, \ldots$ sequence of Borel prob measures on $\mathbb{R}$
(ie $(\mathbb{R}, \mathcal{B}, \mu_i)$ prob triple)
$\exists$ prob triple $(\Omega, \mathcal{F}, P)$ and r.v's on $\mathcal{F}$, r.v's $X_1, X_2, \ldots$
$\ni$ Law$(X_i) = \mathcal{L}(X_i) = \mu_i$ and $X_1, X_2, X_3, \ldots$ are independent.

Lemma 7.1.2

Let U is Uniform r.v. wrt Lebesgue measure
(ie $([0,1], \mathcal{F}, \lambda)$ as constructed in Chapter 2)
$\mathcal{F} = \mathcal{M}$
$U(w) = w$
define for a cdf F (of the form $F(x) = P(X \le x) = P(X \in (-\infty, x])$ for some r.v. X) $(F : \mathbb{R} \to [0,1]$, $F \uparrow$, right continuous$)$
$\phi : (0,1) \to \mathbb{R}$
by $\phi(u) = \inf(x : F(x) \ge u)$
Then $X = \phi(U)$ is a r.v and $P(X \le x) = F(x)$
$\mathcal{L}(X) \equiv F$ in the sense

Proof of Lemma:

$$P(X \le x) = P(U \in \{u \mid \phi(u) \le x\}) = P(U \le F(x))$$
$$\stackrel{?}{=} P(U \in \{u : F(x) \ge u\}) = F(x)$$

$\therefore$ Need to show
$\phi(u) \le x$ iff $F(x) \ge u$
if $y \ge x$, $F(y) \ge F(x)$
$\therefore$ if $x_0 = \phi(u)$, then $F(y) \ge u$ $\forall y \ge \phi(u)$
Take $y \downarrow x_0$ $F(y) \downarrow F(x_0)$ by right continuity
$\therefore x$ has the property $F(x) = \lim_{y \downarrow \phi(u)} F(y) \ge u$ ie $x_0 = \phi(u) = \min(y \mid F(y) \ge u)$
From this it can be shown (by you) that
$\{u \mid \phi(u) \le x\} = \{u \mid F(x) \ge u\}$

Proof of
Theorem 7.1.1
we only need to show $\exists (\Omega, \mathcal{F}, P)$ with r.v's $U_1, U_2, \ldots$ that are iid Unif$(0,1)$

Rosenthal text
$(\Omega, \mathcal{F}, P)$
$\Omega = \prod_{i=1}^{\infty} \{0,1\}$ – set of sequences of 0, 1's See §2.6
domain for his iid coin tossing
$\Omega = \{(r_1, r_2, r_3, \ldots) \mid r_i = 0 \text{ or } 1, i \ge 1\}$
- defined semi-algebra
$\mathcal{J} = \{A_{a_1 a_2 \cdots a_n} \mid n \ge 1, a_1, \ldots, a_n \text{ given values } 0 \text{ or } 1\}$

$$P(A_{a_1 \cdots a_n}) = \left(\tfrac{1}{2}\right)^n = \cdot \; P(\{a_1\} \times \{0,1\} \times \cdots) \cdot P(\{0,1\} \times \{a_2\} \times \{0,1\} \times \cdots) \; \vdots = \prod_{i=1}^{n} \underbrace{P_i(a_i)}\; \prod_{i \ge n+1} \underbrace{P(\{0,1\})}_{=1}$$

by Problem 4.5.15 shows

$= \prod_{i=1} \underbrace{P_i(a_i)}_{\frac{1}{2}}$ (sm?) $\underbrace{\quad}_{=1}$

by $\frac{\text{Problem } 4.5.15}{\text{shows}}$ (2.5.5) holds and therefore Corollary 2.5.4 holds $\exists$ P^* which extends P, $\mathcal{F}$ = smallest σ-alg contain $\mathcal{J}$

$\Rightarrow$ $Y_i(w) = w_i$ are iid fair coin tosses

See p74

$$\begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ \vdots & & \end{pmatrix} = \begin{pmatrix} r_1 & r_3 & r_6 & r_{10} & \cdots \\ r_2 & r_5 & r_9 & & \\ r_4 & r_8 & & & \\ r_7 & & & & \end{pmatrix}$$

$U_i = \sum_{j=1}^{\infty} \frac{1}{2^j} z_{ij}$ ~ Uniform $(0,1)$ i^{th} row

Using Lemma 7.1.2 F_i = cdf of measure μ_i

$X_i = \phi_{F_i}(U_i)$