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Alternate:

recall Sec 2.5 Product Measures (p 22)

$n=2$

$(\Omega_1, \mathcal{F}_1, P_1)$, $(\Omega_2, \mathcal{F}_2, P_2)$ prob triples

Consider

$\Omega = \Omega_1 \times \Omega_2$

$\mathcal{J} = \{ A \times B : A \in \mathcal{F}_1, B \in \mathcal{F}_2 \}$

$\mathcal{J}$ is a semialgebra

$P(A \times B) = P_1(A) \cdot P_2(B)$ - defined on $\mathcal{J}$

Ex. 4.5.15 allows (one way) to verify (2.5.5) of Corollary 2.5.4

$\therefore \exists (\Omega, \mathcal{F}, P^*)$ where $\mathcal{F}$ is a σ-field containing $\mathcal{J}$

and P^* extends P

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$(\Omega_j, \mathcal{F}_j, \mu_j)$ prob triples, r.v.'s X_j on this, $\mathcal{L}aw(X_j) = \mu_j$

$\overset{\shortparallel}{([0,1], \mathcal{B}, \lambda)} \xrightarrow{X_j} ([0,1], \mathcal{B}, \underbrace{\lambda \circ X_j^{-1}}_{=\mu_j})$

$\Omega = \bigtimes_{i=1}^{\infty} \Omega_i$

$\mathcal{J} = \{ A = A_1 \times A_2 \times \dots = \bigtimes_{i=1}^{\infty} A_i \mid$ only a finite number of $A_i \neq \Omega_i \}$

= finite dimensional rectangles

show $\mathcal{J}$ is a semialgebra

$= \{ A = \bigtimes_{i=1}^{\infty} A_i \mid A = A_1 \times \dots \times A_n \times \Omega_{n+1} \times \dots$; for some n, $A_1, \dots, A_n \in \mathcal{F}_1, \dots, \mathcal{F}_n$ respectively $\}$

and for elements of $\mathcal{J}$

$P(A) = P(\bigtimes_{i=1}^{\infty} A_i) = \prod_{i=1}^{\infty} P_i(A_i)$

- this is well defined since $\exists n \ni. \ \forall m > n, A_m = \Omega_m$

so $P(A) = P_1(A) \cdots P_n(A_n) \cdot \prod_{n+1} 1$

Verify (2.5.5) using almost direct analogue of (4.5.15)

Chapter 9. More Limit Theorems

Theorem 9.1.1 (Fatou's Lemma)

If $X_n \geq C \quad \forall n$ (C can be any finite constant)

$E(\liminf X_n) \leq \liminf_{n \to \infty} E(X_n)$

$$E(\liminf_{n\to\infty} X_n) \le \liminf_{n\to\infty} E(X_n)$$

Proof:

$Y_n = \inf_{k\ge n} X_k \uparrow$ Let $Y = \lim_{n\to\infty} Y_n = \lim_{n\to\infty} \inf_{k\ge n} X_k$
$= \liminf_{n\to\infty} X_n$

By Monotone Convergence Theorem (Ch. 4)

$\therefore \lim_{n\to\infty} \underbrace{E(Y_n)} = E(Y) = E(\liminf_{n\to\infty} X_n)$

$LHS = \lim_{n\to\infty} E(\inf_{k\ge n} X_k)$

$\le \lim_{n\to\infty} \inf_{k\ge n} E(X_k)$

$= \liminf_{n\to\infty} E(X_n)$

$Y_n \le X_n$

$E(Y_n) \le E(X_k)\ \forall k\ge n$

$\therefore E(Y_n) \le \inf_{k\ge n} E(X_k)$

$\therefore RHS \le$ ~~R~~ $^{\text{good}}$ LHS

Theorem 9.1.2 (Dominated Convergence Theorem)

$X, X_1, X_2, \dots$ r.v's, $X_n \to X$ with prob. 1

and if $\exists$ r.v. Y with $|X_n| \le Y$ for all n, $E(Y) < \infty$

then

$\lim_{n\to\infty} E(X_n) = E(X)$

Pf.

$Y + X_n \ge 0$ $\quad (|X_n| \le Y \iff \underbrace{-Y \le X_n \le Y})$

$Y - X_n \ge 0$

Apply Fatou's Lemma

$\underline{\liminf_{n\to\infty} E(Y + X_n)} = E(Y) + \liminf_{n\to\infty} E(X_n)$

$\le E(\liminf_{n\to\infty} (Y + X_n))$

$= E(Y + \liminf X_n) = E(Y) + E(\underbrace{\liminf X_n}_{=X})$

$= E(Y) + E(X)$

Also similarly (apply Fatou to $Y - X_n$)

$\therefore E(Y) + \liminf_{n\to\infty} (-E(X_n)) \le E(Y) + (E(-X))$

$\therefore E(Y) - \limsup E(X_n) \le E(Y) - E(X)$

$\therefore E(X) \ge \limsup E(X_n)$

$\limsup E(X_n) \le E(X) \le \liminf E(X_n)$ $\quad \therefore \lim E(X_n)$ exists,

$\therefore \liminf E(X_n) = \limsup E(X_n)$

§9.2 differentiating under $E(\cdot)$

$\frac{d}{dt} \int e^{st} ds \overset{?}{=} \int \frac{d}{dt} e^{st} ds = \int s \cdot e^{st} ds$

set up prop below so Dominated Convergence condition hold

Prop. 9.2.1

$\{F_t\}_{a<t<b}$ family of r.v's defined on some prob triple

exist

$\{F_t\}_{a<t<b}$ family of r.v's defined ...

$F_t'(w) = \frac{\partial}{\partial t} F_t(w)$ exist

Suppose $\exists$ r.v Y, $\ni$ $|F_t'| \le Y$ $\forall^{t\ni}$ $a<t<b$ and $E(Y)<\infty$

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Let $\phi(t) = E(F_t)$.

Then ϕ is differentiable w.r.t and

$\phi'(t) = E(F_t')$ $\forall t \ni a<t<b$.

Pf:

$$\phi'(t) = \lim_{h\to 0} \frac{\phi(t+h)-\phi(t)}{h}$$

$$= \lim_{n\to\infty} \frac{\phi(t+\frac{1}{n}) - \phi(t)}{\frac{1}{n}}$$

- so we can see this is also a limit of a sequence)

$$= \lim_{h\to 0} E\left(\frac{F_{t+h}-F_t}{h}\right)$$

$$= E\left(\lim_{h\to 0}\left(\frac{F_{t+h}-F_t}{h}\right)\right)$$

For each w

$$\lim \frac{F_{t+h}(w) - F_t(w)}{h} = F_t'(w)$$

we next need to show

$$\left|\frac{F_{t+h}(w) - F_t(w)}{h}\right| \le \tilde{Y}(w) = Y(w)+1$$

Given $\epsilon > 0$, $\exists$ $h_0(w,\epsilon)$ $\forall |h| \le h_0$

$$-Y(w)-\epsilon \le F_t'(w) - \epsilon \le \frac{F_{t+h}(w) - F_t(w)}{h} \le F_t'(w)+\epsilon = Y(w)+\epsilon$$

Thus Dominated Convergence (using $\tilde{Y}$) applies

$(E(\tilde{Y}) = E(Y)+1 < \infty)$

$\therefore$ $\frac{\partial}{\partial t} E(F_t) = E\left(\frac{\partial F_t}{\partial t}\right)$

Application to Moment Generating Functions

X r.v., M_X:

$M_X(t) = E(e^{tX})$

defined and finite for all $|t| < s_0$, $s_0 > 0$

Theorem 9.33

X has mgf ie M_X as above)

then $M_X(t) = \sum_{n=0}^{\infty} E(X^n) \cdot \frac{t^n}{n!}$

Proof.

$$Z_n = 1 + t' + \frac{1}{2}(tX)^2 + \dots + \frac{tX^n}{n!}$$

$$\le e^{tX} + e^{-tX} = Y$$

$$E(Y) \le E(e^{tX}) + E(e^{-tX}) = M_X(t) + M_X(-t) < \infty$$

$$Z_n \to \sum_{j=0}^{\infty} \frac{t^j X^j}{j!} = e^{tX}$$

$$(\ ) - \sum \frac{t^j E(X^j)}{j!} \Rightarrow E(Z_j) = M_X(t)$$

$Z_n \to \sum_{j=0} \frac{t \ldots}{j!}$

$\therefore E(Z_n) = \sum_{j=0}^{n} \frac{t^j E(X^j)}{j!} \to E(Z) = M_X(t)$