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Fubini's Theorem

μ, ν measures on σ-field for $\mathcal{X}, \mathcal{Y}$ respectively

$\mu \times \nu$

$(\underline{\mathcal{X} \times \mathcal{Y}}, \sigma(\underline{\sigma(\mathcal{X})} \times \underline{\sigma(\mathcal{Y})}), \mu \times \nu)$

$$\iint_{\mathcal{X} \times \mathcal{Y}} f \, d(\mu \times \nu)$$
$$= \int_{\mathcal{X}} \left\{ \int_{\mathcal{Y}} f(x,y) \nu(dy) \right\} \mu(dx)$$
$$= \int_{\mathcal{Y}} \left\{ \int_{\mathcal{X}} f(x,y) \mu(dx) \right\} \nu(dy)$$

* If f is $(\mathcal{X} \times \mathcal{Y}, \sigma(\sigma(\mathcal{X}) \times \sigma(\mathcal{Y})))$ measureable

and $\iint_{\mathcal{X} \times \mathcal{Y}} |f| \, d(\mu \times \nu) < \infty$

or $f \geq C > -\infty$, $\iint f d\mu \times \nu = +\infty$ (if dealing with prob measures)

or $f \leq C < \infty$, $\iint f d\mu \times \nu = -\infty$ (")

Exercise 9.5.13 $\leftarrow$ product counting measure

9.5.14 $\leftarrow$ product Lebesgue measure on $(0,1) \times (0,1)$

Aside: μ counting measure on $\mathbb{N}_0$ (or prob. measure on $\mathbb{N}_0$)

$$\int f \, d\mu = \sum_x f(x) \mu(dx) = \sum_{n=0}^{\infty} f(n) \mu(dn)$$

Proof:

standard idea (1) f simple.

(2) $f \geq 0$, f_n simple, $f_n \uparrow f$

(3) $f = f^+ - f^-$, extend (2) to this case

-here need f^+, f^- to be $\mathcal{X} \times \mathcal{Y}$ finite integrable

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Chapter 10

Weak Convergence (Convergence in Distribution)

Definition $\{\mu_n\}$ sequence of probability measures converges weakly to μ iff for all continuous f

$\Rightarrow \mu$

$\int f \, d$

converges weakly,
continuous f

$\mu_n \Rightarrow \mu$ notation for weak convergence

$$\int f \, d\mu_n \to \int f \, d\mu$$

In Rosenthal, we next specialize to μ_n being probability measures on $(\mathbb{R}, \mathcal{B})$

§10.1 Equivalent Conditions (Characterizations..) of Weak Convergence

Theorem 10.1.1 The following are equivalent

(a) $\mu_n \Rightarrow \mu$

(b) $\mu_n(A) \to \mu(A)$ for all A such that $\mu(\partial A) = 0$, where ∂A = boundary of A

(c) $\mu_n((-\infty, x]) \to \mu((-\infty, x])$ for all x for which $\mu(\{x\}) = 0$

(d) [Skorokhod's Theorem])
$\exists$ prob triple, r.v.s $Y, Y_1, Y_2, Y_3, \ldots \ni$
$\mathcal{L}aw(Y_n) = \mu_n$, $\mathcal{L}aw(Y) = \mu$
and $Y_n \to Y$ a.s.

(e) $\int_{\mathbb{R}} f \, d\mu_n \to \int f \, d\mu$ for all bounded Borel measurable $f \ni \mu(D_f) = 0$
where
$D_f = \{ x \mid f \text{ is not continuous at } x \}$
= set of f-discontinuities